A Concise Course in Arakelov Geometry

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Preface

The rudiment of Arakelov's theory comes from discussing the volume of lattices of number fields, using Minkowski's theory. By some non-geometric technologies, we can prove the Serre duality theorem and the Riemann-Roch theorem of a ring of algebraic integers (geometrically speaking, this is a 1-dimensional curve), but these formulas come from algebraic geometry. This implies that our algebraic number theory should have a geometric interpretation, and its high-dimensional version should contain more abundant arithmetic information. This geometry was later called Arakelov geometry.

The serious Arakelov theory of surfaces was established by Arakelov [Arak] in 1974. It is a kind of intersection theory connecting algebraic geometry and complex geometry. Then, in 1984, Faltings gave an arithmetic Riemann-Roch formula using Arakelov's theory [Falt], this showed that Arakelov geometry is powerful.

In 1990, Gillet and Soulé developed the Arakelov intersection theory on general arithmetic varieties [GiS1], and in 1992, they extended the arithmetic Riemann-Roch formula by using this theory [GiS2]. In 2008, their subsequent work with Rössler [GRS], proved the formula in the case of higher degrees.

We assume that the readers are familiar with algebraic geometry, differential geometry and algebraic number theory. Although this note does not presuppose knowledge of complex geometry, it is better if you master it.

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Chapter 1

Curves and Number Theory

In this chapter, we introduce the so-called "geometry of numbers", which studies fractional ideals of number fields by embedding them into the field of complex numbers. One of the most important theorems in this theory is the Riemann-Roch formula for arithmetic varieties.

1.1 Fractional Ideals and Invertible Sheaves

Let K be a number field, write $X := \mathrm{Spec}(\mathscr{O}_K)$. There are three ways to study the primes in $\mathscr{O}_K$:

- Number theoretically. A **fractional ideal** of K is a $\mathscr{O}_K$-submodule of K of rank 1, a **principal fractional ideal** of K is a fractional ideal has form $x\mathscr{O}_K$ for some $x \in K^\times$. Define the **ideal class group** of K to be

$$\mathrm{Cl}(\mathscr{O}_K) := \{\text{fractional ideals of } K\}/\{\text{principal fractional ideals of } K\},$$

  with the usual multiplication of ideals. The inverse of a fractional ideal $\mathfrak{a}$ in this group is $\mathfrak{a}^{-1} := \{x \in K : x\mathfrak{a} \subseteq \mathscr{O}_K\}$.

- Geometrically. Consider the scheme X, by a **invertible sheaf** (or a **line bundle**) we mean a rank 1 locally free $\mathscr{O}_X$-module on X. For a scheme X we can define a **Picard group**

$$\mathrm{Pic}(X) := \text{isomorphism classes of invertible sheaves on } X,$$

  with the multiplication given by tensor product. The inverse of a invertible sheaf is obtained by dualizing.

- Geometrically. Consider the scheme X. A **divisor** on X is a codimension 1 subscheme of X. Since X has Krull dimension 1, a divisor must be a finite formal sum of some closed points in X. A **principal divisor** on X is a divisor has form $\sum_{\mathfrak{p}} \mathrm{ord}_{\mathfrak{p}}(x)\,\mathfrak{p}$ for some $x \in K^\times$. Define the **divisor class group**

$$\mathrm{CH}^1(X) := \{\text{divisors on } X\}/\{\text{principal divisors on } X\},$$

with the addition given by the formal sum. The inverse of a divisor is added a minus sign.

Algebraic geometry tells us the second and third methods are essentially the same. That is, studying the codimension 1 closed subschemes is equivalent to studying the line bundles. Indeed, the above three methods are all equivalent.

Proposition 1.1.1. *There are isomorphisms*

$$\mathrm{Cl}(\mathscr{O}_K) \cong \mathrm{Pic}(X) \cong \mathrm{CH}^1(X) \cong K^\times \Big\backslash \mathbb{A}_K^\times \Big/ \widehat{\mathscr{O}_K^\times} K_\infty^\times,$$

where $\mathbb{A}_K^\times$ is the group of units in the ring of adeles of K, and

$$\widehat{\mathscr{O}_K^\times} := \prod_{v \text{ finite}} \mathscr{O}_{K_v}^\times \times \prod_{v \text{ infinite}} \{1\}, \quad K_\infty^\times := \prod_{v \text{ finite}} \{1\} \times \prod_{v \text{ infinite}} K_v^\times.$$

Proof. We only give the definitions of these maps.

- Since X is affine, we have $\mathrm{Cl}(\mathscr{O}_K) \cong \mathrm{Pic}(X)$.
- $\mathrm{Cl}(\mathscr{O}_K) \cong \mathrm{CH}^1(X)$ is given by $\prod_{\mathfrak{p}} \mathfrak{p}^{-n_\mathfrak{p}} \leftrightarrow \sum_{\mathfrak{p}} n_\mathfrak{p}\, \mathfrak{p}$.
- The map $K^\times \Big\backslash \mathbb{A}_K^\times \Big/ \widehat{\mathscr{O}_K^\times} K_\infty^\times \cong \mathrm{Cl}(\mathscr{O}_K)$ is given by $(x_v)_v \leftrightarrow \prod_{v \text{ finite}} \mathfrak{p}_v^{n_v}$, where $\mathfrak{p}_v$ is the prime ideal corresponding to v and n_v is the v-adic valuation of x_v.

The remaining work is for you. □

Remark 1.1.2. The map from the Picard group $\mathrm{Pic}(X)$ to the divisor class group $\mathrm{CH}^1(X)$ is called the **first Chern class**, this map is given by taking "zeros minus poles" of some rational global section of a invertible sheaf. Its inverse $D \mapsto \mathscr{O}_X(D)$ can be defined as we will do in Chapter 2.1.

Recall we have the product formula

$$\prod_{v \text{ finite}} |x|_v = \prod_{v \text{ infinite}} |x|_v^{-1}$$

for all $x \in K^\times$. It implies that, in the premise of satisfying this formula, in order to obtain a "compactification" of X in a certain sense, some information at infinite places can be added to the divisors defined previously. This is the original idea of Arakelov's theory.

Definition 1.1.3. Let us make some definitions in parallel.

- Number theoretically. We use the adelic version of $\mathrm{Cl}(\mathscr{O}_K)$ to generalize the definition. That is, we define the **Arakelov class group** to be the locally compact group
$$\widehat{\mathrm{Cl}}(\mathscr{O}_K) := K^\times \Big\backslash \mathbb{A}_K^\times \Big/ \widehat{\mathscr{O}_K^\times}\, \mathscr{O}_\infty^\times,$$
where $\mathscr{O}_\infty^\times$ is the maximal compact subgroup of $K_\infty^\times$.

- Geometrically. Consider the scheme X. A **metrized invertible sheaf** $(\mathscr{L}, \|\cdot\|_\tau)$ (or a **metrized line bundle**) is a invertible sheaf $\mathscr{L}$ together with a collection of non-trivial norms (hence induce Hermitian inner products) $\|\cdot\|_\tau$ on 1-dimensional complex linear spaces $\mathscr{L} \otimes_\tau \mathbb{C}$ for each embedding $\tau \in \mathrm{Hom}(K, \mathbb{C})$, invariant under the action of complex conjugation. Define the **arithmetic Picard group** of X to be

$$\widehat{\mathrm{Pic}}(X) := \text{isometry classes of metrized line bundles on } X.$$

- Geometrically. Consider the scheme X. An **Arakelov divisor** on X is an element in the group

$$\widehat{Z}^1(X) := \{\text{divisors on } X\} \oplus \left(\bigoplus_{\tau \in \mathrm{Hom}(K,\mathbb{C})} \mathbb{R} \right)^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})},$$

where $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts by $\tau \to \overline{\tau}$. Inside this group, we form a subgroup

$$\widehat{R}^1(X) := \left\{ \widehat{\mathrm{div}}(x) := \Big(\sum_{\mathfrak{p}} \mathrm{ord}_{\mathfrak{p}}(x)\, \mathfrak{p}, (-\log|\tau(x)|^2)_\tau \Big) : x \in K^\times \right\}.$$

The quotient group $\widehat{\mathrm{CH}}^1(X) := \widehat{Z}^1(X)/\widehat{R}^1(X)$ is called the **arithmetic Chow group**.

For the Arakelov case, we have a conclusion similar to Proposition 1.1.1.

Proposition 1.1.4. *There are isomorphisms*

$$\widehat{\mathrm{Cl}}(\mathscr{O}_K) \cong \widehat{\mathrm{Pic}}(X) \cong \widehat{\mathrm{CH}}^1(X).$$

Proof. We still only give the definitions of the maps.

- The isomorphism $\widehat{\mathrm{Pic}}(X) \xrightarrow{\sim} \widehat{\mathrm{CH}}^1(X)$ is given by

$$(\mathscr{L}, \|\cdot\|_\tau) \mapsto \widehat{\mathrm{div}}(s) := \left(\sum_{\mathfrak{p}} n_{\mathfrak{p}}\, \mathfrak{p}, (-\log\|s_\tau\|_\tau^2)_\tau \right)$$

for some rational global section s of $\mathscr{L}$, where $s_\tau \in \mathscr{L} \otimes_\tau \mathbb{C}$ is the pull-back of s by τ, and $n_{\mathfrak{p}}$ is the order of vanishing of s at $\mathfrak{p}$.

- To construct an isomorphism from $\widehat{\mathrm{Pic}}(X)$ to $\widehat{\mathrm{Cl}}(\mathscr{O}_K)$, suppose we have a metrized line bundle $(\mathscr{L}, \|\cdot\|_\tau)$ on X. Choose a rational section s of $\mathscr{L}$, we associate s an idele

$$\Big((\varpi_v^{n_v})_{v \text{ finite}}, (\|s_\tau\|_\tau)_{\tau \text{ infinite}} \Big),$$

where ϖ_v is a uniformizer of K_v and n_v is the order of vanishing of s at v.

The remaining work is for you. □

The map from $\widehat{\mathrm{Pic}}(X)$ to $\widehat{\mathrm{CH}}^1(X)$ is called the **first arithmetic Chern class**.

If we take $K = \mathbb{Q}$ in Definition 1.1.3, then $\widehat{\mathrm{Cl}}(\mathbb{Z})$ is actually a **Shimura variety** relative to the Shimura datum $(\mathrm{GL}_1, \{\mathrm{pt}\})$ with arithmetic subgroup $\{\pm 1\}$. Moreover, there is a natural isomorphism

$$\mathrm{Log} : \widehat{\mathrm{Cl}}(\mathbb{Z}) = \mathbb{Q}^\times \Big\backslash \mathbb{A}_\mathbb{Q}^\times \Big/ \left(\widehat{\mathbb{Z}}^\times \times \{\pm 1\}\right) \xrightarrow{\sim} \mathbb{R}, \quad (x_v)_v \longmapsto \sum_{p<\infty} \mathrm{ord}_p(x_p) \log p - \log |x_\infty|.$$

This map induces maps $\widehat{\mathrm{CH}}^1(\mathrm{Spec}(\mathbb{Z})) \to \mathbb{R}$ and $\widehat{\mathrm{Pic}}(\mathrm{Spec}(\mathbb{Z})) \to \mathbb{R}$ by Proposition 1.1.4. Based on this, we can define the most important invariant of metrized line bundles: the Arakelov degree map (here just only provide the definition in the case of curves, and of course, it can be generalized to higher dimensions using push-forward and the arithmetic Riemann-Roch formula).

Definition 1.1.5 (Arithmetic Degree)**.** For a metrized line bundle $(\mathscr{L}, \|\cdot\|_\tau)$ on X, then $\mathscr{L}$ is a fractional ideal of $\mathscr{O}_K$. Take $0 \neq s \in \mathscr{L}$, define

$$\widehat{\deg}(\mathscr{L}, \|\cdot\|_\tau) := \log \#(\mathscr{L}/s \cdot \mathscr{O}_K) - \sum_{\tau \in \mathrm{Hom}(K,\mathbb{C})} \log \|s_\tau\|_\tau \in \mathbb{R},$$

where $s_\tau \in \mathscr{L} \otimes_\tau \mathbb{C}$ is the pull-back of s by τ. By product formula, this definition is independent of the choice of s.

Exercise 1.1.6. *Verify the arithmetic degree* $\widehat{\deg} : \widehat{\mathrm{CH}}^1(\mathrm{Spec}(\mathbb{Z})) \to \mathbb{R}$ *is given by*

$$\left(\sum_p n_p[p], n_\infty\right) \mapsto \sum_p n_p \log p + \frac{n_\infty}{2}.$$

1.2 Riemann-Roch Theorem

In this section we will show the proof of the arithmetic Riemann-Roch formula for curves.

Let K be a number field and $\mathscr{O}_K$ be its ring of integers, write $X := \mathrm{Spec}(\mathscr{O}_K)$, define $K_\mathbb{R} := \left(\prod_{\tau \in \mathrm{Hom}(K,\mathbb{C})} \mathbb{C}\right)^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} \cong K \otimes_\mathbb{Q} \mathbb{R}$.

Fix an Arakelov divisor $\left(\sum_\mathfrak{p} n_\mathfrak{p}\, \mathfrak{p}, (r_\tau)_\tau\right)$. By Proposition 1.1.4, it corresponds to a metrized line bundle

$$\mathscr{L} = \left(\prod_\mathfrak{p} \mathfrak{p}^{-n_\mathfrak{p}}, \left(\|1_\tau\|_\tau = e^{-\frac{1}{2} r_\tau}\right)_\tau\right).$$

Put $\mathscr{L}_{\mathrm{fin}} := \left(\prod_\mathfrak{p} \mathfrak{p}^{-n_\mathfrak{p}}, (1)_\tau\right)$ and $\mathscr{L}_{\mathrm{inf}} := \left((1), (e^{-r_\tau/2})_\tau\right)$. Define a map

$$j : K \to K_\mathbb{R}, \quad x \mapsto (\tau(x))_\tau,$$

then $j(\mathscr{L}_{\mathrm{fin}})$ is a lattice in $K_\mathbb{R}$. On the other hand, $\mathscr{L}_{\mathrm{inf}}$ induces a linear map

$$\rho_\mathscr{L} : K_\mathbb{R} \to K_\mathbb{R}, \quad (s_\tau)_\tau \mapsto \left(e^{-r_\tau/2} s_\tau\right)_\tau$$

of $\mathbb{R}$-linear spaces.

Definition 1.2.1 (Characteristic)**.** Define a map

$$\widehat{\chi}:\widehat{\mathrm{Pic}}(X)\to\mathbb{R},\quad \mathscr{L}\mapsto -\log\big(\mathrm{vol}\big(K_{\mathbb{R}}/(\rho_{\mathscr{L}}\circ j(\mathscr{L}_{\mathrm{fin}}))\big)\big),$$

call it the **arithmetic Euler characteristic**.

Exercise 1.2.2. *Verify this definition is well-defined, i.e. $\widehat{\chi}$ depends only on the class in $\widehat{Z}^1(X)$.*

Let $\mathscr{O}_X=\big((1),(1)_\tau\big)$ be the trivial invertible sheaf with the standard metric. The corresponding Arakelov divisor is $\big((0),(0)_\tau\big)$.

Lemma 1.2.3. $\widehat{\chi}(\mathscr{O}_X)=-\log\sqrt{|\mathrm{disc}(K/\mathbb{Q})|}$.

Proof. Let $e_1,\cdots,e_n$ be an integral basis, then the lattice $j(\mathscr{O}_{X,\mathrm{fin}})$ is spanned by the vectors $(\tau_1(e_j),\cdots,\tau_n(e_j))$ for $j=1,\cdots,n$, where $\tau_i\in\mathrm{Hom}(K,\mathbb{C})$ are embeddings. One can compute the volume is $\mathrm{vol}(K_{\mathbb{R}}/j(\mathscr{O}_{X,\mathrm{fin}}))=|\det(\tau_i(e_j))_{ij}|=\sqrt{|\mathrm{disc}(K/\mathbb{Q})|}$. □

Now we obtain the most important formula in this section:

Theorem 1.2.4 (Arithmetic Riemann-Roch)**.** *For any $\mathscr{L}\in\widehat{\mathrm{Pic}}(X)$, we have*

$$\widehat{\chi}(\mathscr{L})-\widehat{\chi}(\mathscr{O}_X)=\widehat{\deg}(\mathscr{L}).$$

Proof. Suppose $\mathscr{L}=\big(\prod_{\mathfrak{p}}\mathfrak{p}^{-n_{\mathfrak{p}}},(e^{-r_\tau/2})_\tau\big)$, then

$$\widehat{\deg}(\mathscr{L})=\sum_{\mathfrak{p}}n_{\mathfrak{p}}\log\#k(\mathfrak{p})+\sum_{\tau}\frac{r_\tau}{2}.$$

Consider the sublattice $j(\mathscr{L}_{\mathrm{fin}})$ of $j(\mathscr{O}_{X,\mathrm{fin}})$, it defines a linear endomorphism $\Theta:K_{\mathbb{R}}\to K_{\mathbb{R}}$ such that $\Theta\circ j(\mathscr{O}_{X,\mathrm{fin}})=j(\mathscr{L}_{\mathrm{fin}})$. Thus

$$|\det(\Theta)|=[j(\mathscr{O}_{X,\mathrm{fin}}):j(\mathscr{L}_{\mathrm{fin}})]=\Big[\mathscr{O}_K:\prod_{\mathfrak{p}}\mathfrak{p}^{-n_{\mathfrak{p}}}\Big]=\prod_{\mathfrak{p}}\#k(\mathfrak{p})^{-n_{\mathfrak{p}}}.$$

Using this, we compute

$$\begin{aligned}\widehat{\chi}(\mathscr{L})&=-\log\big(\mathrm{vol}\big(K_{\mathbb{R}}/(\rho_{\mathscr{L}}\circ j(\mathscr{L}_{\mathrm{fin}}))\big)\big)\\&=-\log\big(\mathrm{vol}\big(K_{\mathbb{R}}/(\rho_{\mathscr{L}}\circ\Theta\circ j(\mathscr{O}_{X,\mathrm{fin}}))\big)\big)\\&=-\log\big(\det(\rho_{\mathscr{L}})\cdot\det(\Theta)\cdot\mathrm{vol}(K_{\mathbb{R}}/j(\mathscr{O}_{X,\mathrm{fin}}))\big)\\&=\sum_{\tau}\frac{r_\tau}{2}+\sum_{\mathfrak{p}}n_{\mathfrak{p}}\log\#k(\mathfrak{p})+\widehat{\chi}(\mathscr{O}_X),\end{aligned}$$

as desired. □

Remark 1.2.5. The arithmetic Riemann-Roch formula also studies the behavior of the push-forward of a metrized line bundle, or equivalently speaking, compute how push-forward affects the arithmetic degree. The so-called push-forward is the map

$$\pi_*:\widehat{\mathrm{Pic}}(X)\to\widehat{\mathrm{Pic}}(\mathrm{Spec}(\mathbb{Z}))\text{ or }\pi_*:\widehat{\mathrm{CH}}^1(X)\to\widehat{\mathrm{CH}}^1(\mathrm{Spec}(\mathbb{Z}))$$

induced by $\pi : X \to \mathrm{Spec}(\mathbb{Z})$, which is coincide with the norm map in algebraic number theory.

Indeed, for a metrized line bundle $\mathscr{L}$ on $\mathrm{Spec}(\mathscr{O}_K)$, there is an equality

$$\widehat{\deg}(\pi_* \mathscr{L}) = \widehat{\deg}(\mathscr{L}) - \log \sqrt{|\mathrm{disc}(K/\mathbb{Q})|}.$$

So there is no reason why $\widehat{\deg}$ and π_* should commute, the commutativity issue is a well-known problem in algebraic geometry: Grothendieck-Riemann-Roch Theorem.

Chapter 2

Surfaces and Arakelov Theory

In this chapter, we introduce the Arakelov theory of surfaces developed by Arakelov and Faltings.

2.1 Riemann Surfaces

Let X be a compact Riemann surface.

Definition 2.1.1 (Weil Functions)**.** Let $D = \{(U, f)\}$ be a Cartier divisor on X. A **Weil function** associated with D is a map

$$\lambda_D : X \setminus \operatorname{Supp}(D) \to \mathbb{R},$$

such that for every $P \in U \setminus \operatorname{Supp}(D)$, $\lambda_D(P) = -\log|f(P)| + \alpha(P)$ for some smooth function $\alpha : U \to \mathbb{R}$.

The function α here will be viewed as a metric. Note that $\lambda_D(P) = \infty$ is not well-defined when $P \in \operatorname{Supp}(D)$.

Definition 2.1.2 (Néron Functions)**.** Let $D = \{(U, f)\}$ be a Cartier divisor on X, consider the triple (U, f, α) where $\alpha : U \to \mathbb{R}$ is smooth. We say two triples (U, f, α) and (V, g, β) are compatible, if

- $(U, f), (V, g) \in D$. This implies $f/g \in \mathscr{O}_X(U \cap V)^\times$.
- $-\log|f/g| = \beta - \alpha$ holds on $U \cap V$.

A maximal family of compatible triples is called a **Néron divisor**, denoted by $D = \{(U, f, \alpha)\}$. All Néron divisors form an abelian group via

$$(U, f, \alpha) \cdot (V, g, \beta) := (U \cap V, (fg)|_{U \cap V}, (\alpha + \beta)|_{U \cap V}).$$

For a Néron divisor $D = \{(U, f, \alpha)\}$, define the Weil function associated with D to be

$$\lambda_D(P) := -\log|f(P)| + \alpha(P), \quad P \in U \setminus \operatorname{Supp}(D).$$

Néron divisors can be viewed as "metrized" Cartier divisors.

Exercise 2.1.3. *λ_D is independent of the choice of triple.*

Recall that there is a natural way to identify line bundles (or invertible sheaves) on X with Cartier (or Weil) divisors on X. Our goal is to make this correspondence metrically. Let us first review some geometric operations.

Let $\mathscr{L}$ be a line bundle on X, i.e. $X = \bigcup_i U_i$ such that for each i, $\phi_i : \mathscr{L}|_{U_i} \xrightarrow{\sim} \mathscr{O}_X|_{U_i}$ is an isomorphism of $\mathscr{O}_X|_{U_i}$-module, and satisfies

$$\begin{array}{ccc} \mathscr{L}|_{U_i\cap U_j} & \xrightarrow{\quad \mathrm{id} \quad} & \mathscr{L}|_{U_i\cap U_j} \\ \downarrow{\scriptstyle \phi_j} & & \downarrow{\scriptstyle \phi_i} \\ \mathscr{O}_X|_{U_i\cap U_j} & \xrightarrow[\phi_{ij}:=\phi_i\circ\phi_j^{-1}\in\mathscr{O}_X(U_i\cap U_j)^\times]{\sim} & \mathscr{O}_X|_{U_i\cap U_j} \end{array}$$

These functions ϕ_i are called **trivialization functions**. In particular, if $D = \{(U_i, f_i)\}$ is a Cartier divisor, then the line bundle associated with D

$$\mathscr{O}_X(D)(U) := \{f \in \mathscr{M}(U) : \mathrm{div}(f) + D \geq 0\}, \quad U \subseteq X$$

has a well-known trivialization

$$\big(U_i, f_i \times (\cdot) : \mathscr{O}_X(D)|_{U_i} \xrightarrow{\sim} \mathscr{O}_X|_{U_i}\big).$$

Definition 2.1.4 (Metrics)**.** Let $\mathscr{L}$ be a line bundle on X and has trivialization $\mathscr{L} = \{(U, \phi)\}$. Let $h : U \to \mathbb{R}_{>0}$ be smooth functions, consider the triples (U, ϕ, h). We say two triples (U, ϕ, h) and (V, ψ, m) are compatible, if

$$h(P) = |\phi \circ \psi^{-1}(P)|^2 \cdot m(P), \quad P \in U \cap V.$$

A maximal family of compatible triples is called a **metric** on $\mathscr{L}$, denoted by $(\mathscr{L}, h)$. We also call it a **metrized line bundle**.

Remark 2.1.5. Let $\mathscr{L} = \{(U, \phi)\}$ be a line bundle on X. Let $s \in \Gamma(U, \mathscr{L})$ be a section, for $P \in U$, define a norm (hence induces a Hermitian inner product) on the one dimensional $\mathbb{C}$-linear space $\mathscr{L}_P$ (i.e. the fiber of $\mathscr{L}$ at P) to be

$$\|s(P)\|_h := \frac{|\phi_P(s(P))|}{\sqrt{h(P)}}.$$

This number does not depend on the choice of trivialization. It is easy to see that h and $\|\cdot\|_h$ are determine each other, so we will abuse them.

Remark 2.1.6. There are many ways to construct new metrized line bundles.

- Let $(\mathscr{L}, h) = \{(U, \phi, h)\}$ be a metrized line bundle on X, define the **dual bundle** $(\mathscr{L}^{-1}, h^{-1}) := \{(U, \phi^{-1}, h^{-1})\}$.
- Let $(\mathscr{L}, h) = \{(U, \phi, h)\}$, $(\mathscr{M}, m) = \{(U, \psi, m)\}$ be metrized line bundles on X, define the **tensor bundle** $(\mathscr{L} \otimes \mathscr{M}, h \cdot m) := \{(U, \phi \cdot \psi, h \cdot m)\}$.

- Let $f: X \to Y$ be a morphism and $(\mathscr{L}, h) = \{(U, \phi, h)\}$ be a metrized line bundle on Y, then the **pull-back bundle** $f^*\mathscr{L}$ has a metric defined by $(f^*\mathscr{L}, h \circ f) = \{(f^{-1}U, f^\sharp \circ \phi, h \circ f)\}$, where $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$ and the trivialization $f^*\mathscr{L} \to \mathscr{O}_X$ comes from the adjoint pair (f^*, f_*).

Proposition 2.1.7. *Let $D = \{(U, f)\}$ be a Cartier divisor on X, then there is a one-to-one correspondence:*

$$\{\textit{metrics on } \mathscr{O}_X(D)\} \longleftrightarrow \{\textit{Weil functions associated with } D\},$$

given by $h \mapsto (-\log|f| + \frac{1}{2}\log h)$; $(\lambda_D = -\log|f| + \alpha) \mapsto e^{2\alpha}$.

Proof. Let $s = (U, s|_U \in \mathscr{M}(U))$ be a meromorphic global section of $\mathscr{O}_X(D)$ such that $\operatorname{div}(s) = D$, we already know there is a natural trivialization

$$f \times (\cdot) : \mathscr{O}_X(D)|_U \xrightarrow{\sim} \mathscr{O}_X|_U, \quad s|_U \mapsto f \cdot s|_U.$$

For a metric h on $\mathscr{O}_X(D)$, define a function associated with s by

$$\lambda_{h,s}(P) := -\log\|s(P)\|_h = -\log\frac{|f(P)\cdot s|_U(P)|}{\sqrt{h(P)}} = -\log|f(P)\cdot s|_U(P)| + \frac{1}{2}\log h(P).$$

Now take $s = 1_D$ and suppose $(\operatorname{Supp}(D)^c, 1) \in D$, then $\log|f(P)\cdot s|_U(P)|$ vanishes, one can verify the bijection easily. $\square$

In fact, the metric on a vector bundle reflects some geometrical and topological information of this bundle. We now introduce the Chern form of a metric, which can be viewed as an important characteristic class in the cohomology group. This class can be obtained from curvature in differential geometry.

But first, let us recall some notations. Let $z = x + iy$ be a local complex coordinate. Define the differential operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right); \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

For a smooth function f, define $\partial f := \frac{\partial f}{\partial z}dz \in \mathscr{A}^{1,0}$, $\overline{\partial} f := \frac{\partial f}{\partial \bar{z}}d\bar{z} \in \mathscr{A}^{0,1}$ and $d := \partial + \overline{\partial} \in \mathscr{A}^1$, $d^c := \frac{1}{4\pi i}(\partial - \overline{\partial}) \in \mathscr{A}^1$.

Exercise 2.1.8. *Prove:*

$$\partial\overline{\partial} = -\overline{\partial}\partial = -2\pi i dd^c = -\frac{i}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) dx \wedge dy \in \mathscr{A}^{1,1}.$$

Remark 2.1.9. This is a warning. Let (X, g) be a n dimensional projective complex manifold with Kähler metric g and its volume form $\mathrm{vol}_g \in \mathscr{A}^{2n}$. It is unreasonable to use Exercise 2.1.8 to define the Laplacian on X, since $\sum_{i=1}^{2n} \frac{\partial^2}{\partial x_i^2}$ can not carry metric information and may not glue into a global operator. But from linear algebra, g induces a Hermitian inner product $\widetilde{g}$ on $\bigwedge^k T^*X$, the space of k-forms, $0 \le k \le 2n$. Now define the L^2-scalar product

$$\langle \cdot, \cdot \rangle_g : \mathscr{A}^k \times \mathscr{A}^k \to \mathbb{C}, \quad \langle \omega, \eta \rangle_g \mapsto \int_X \widetilde{g}(\omega, \eta) \cdot \mathrm{vol}_g.$$

If we write the right adjoint of d for $\langle\cdot,\cdot\rangle_g$ as d^*, one can define the Laplacian $\Delta_{\mathrm{dR}} := dd^* + d^*d$, called the **Laplace-de Rham operator**. In Euclidean plane, this Laplacian coincides with the ordinary one (up to a sign).

Definition 2.1.10 (Chern Forms)**.** Let $(\mathscr{L},h) = \{(U,\phi,h)\}$ be a metrized line bundle on X, let s be a holomorphic section on U. Define the **Chern form** of $(\mathscr{L},h)$ to be

$$c_1(\mathscr{L},h) := dd^c \log h(z) = -dd^c \log \|s(z)\|_h^2, \quad z \in U \setminus \mathrm{Supp}(\mathrm{div}(s)).$$

Its cohomology class in the de Rham cohomology group $H^2_{\mathrm{dR}}(X)$ is called the **Chern class**, also denoted by $c_1(\mathscr{L},h)$.

Since the transition functions are holomorphic non-zero, it follows that one can glue $c_1(\mathscr{L},h)$ into a global form in $\mathscr{A}^{1,1}$.

Remark 2.1.11. In complex geometry, let E be a Hermitian vector bundle on a complex manifold X. For each $P \in X$, the fiber E_P is a finite dimensional $\mathbb{C}$-linear space and has a Hermitian inner product

$$\langle\cdot,\cdot\rangle_P : E_P \times E_P \to \mathbb{C}.$$

Suppose E has a frame $\{e_i\}$ composed of global sections. There are some important matrices:

- The metric matrix $H := [\langle e_i, e_j\rangle] \in \mathrm{Mat}_n^{0\text{-form}}$. It is not hard to see $H = \overline{H}^T$.
- The connection matrix $W \in \mathrm{Mat}_n^{1\text{-form}}$. Let

  $$\nabla : \Gamma(U,E) \to \Gamma(U, T^*X \otimes E) \cong \Gamma(U, \mathrm{Hom}(TX,E))$$

  be the connection induced by H, then $W := [w_{ij}]$ is defined by $\nabla e_j = \sum_{i=1}^n w_{ij} e_i$. One can show that $W = \overline{H}^{-1}\partial(\overline{H})$.
- The curvature matrix $\Omega = dW + W \wedge W = \overline{\partial}(\partial(H)\cdot H^{-1})^T \in \mathrm{Mat}_n^{2\text{-form}}$, by Bianchi identity.

If E is a line bundle, then $\Omega = -\partial\overline{\partial}(\log H)$. This explains why we define Chern forms in a strange expression.

Proposition 2.1.12. *Let $(\mathscr{L},h)$ be a metrized line bundle on X, then*

$$\int_X c_1(\mathscr{L},h) = \deg(\mathscr{L}).$$

Proof. Let s be a meromorphic section, so $c_1(\mathscr{L},h) = -dd^c \log \|s(z)\|_h^2$ outside the support of $\mathrm{div}(s)$. At each point P where s has a zero or pole, we pute a small circle $C(P,r)$ of radius r. Represent $\|s(z)\|_h^2 = f\overline{f}g$ where f is meromorphic at P and g is smooth positive, apply Stokes' formula we have

$$\int_X c_1(\mathscr{L},h) = \lim_{r\to 0}\sum_P \int_{C(P,r),\curvearrowleft} d^c \log\|s(z)\|_h^2 = \lim_{r\to 0}\sum_P \int_{C(P,r),\curvearrowleft} \frac{\partial - \overline{\partial}}{4\pi i}(\log f + \log\overline{f} + \log g).$$

the $\log g$ term is bounded locally so the integral of this term tends to 0. Obviously, $\overline{\partial}\log f = \partial \log \overline{f} = 0$, so the integral becomes

$$\int_X c_1(\mathscr{L},h) = \frac{1}{4\pi i}\lim_{r\to 0}\sum_P 2i\mathrm{Im}\int_{C(P,r),\curvearrowleft}\frac{f'}{f}dz = \sum_P \mathrm{ord}_P(f) = \deg(\mathscr{L}),$$

as desired. □

Exercise 2.1.13. *Let* $[x:y]\in\mathbb{P}^1(\mathbb{C})$*, define a metric*

$$h:\mathbb{C}^\times \to \mathbb{R}_{>0}, \quad [x:y]\mapsto |x|^2+|y|^2.$$

- *If* $x\neq 0$*, then* $z := [x:y] = [1:\frac{y}{x}]\in\mathbb{C}$*. Show the metric* h *in this coordinate is* $z\mapsto 1+|z|^2$*.*
- *Find the line bundle where* h *lives in.*
- *If we consider* $dd^c\log h\in\mathscr{A}^{1,1}$ *as a measure, this defines a Hermitian metric on* $\mathbb{P}^1(\mathbb{C})$*, called the* ***Fubini-Study metric****. Find the curvature of this metric and which line bundle where the metric lives in (recall the degree of holomorphic tangent bundle on* $\mathbb{P}^1(\mathbb{C})$ *is* 2*).*

2.2 Green Functions and Metrics

Let X be a compact Riemann surface. In this section, we study a special case of Weil functions on X, which are Green functions.

If the genus g of X is bigger than 0, define the **canonical volume form** on X to be

$$\mu := \frac{i}{2g}\sum_{k=1}^{g}\omega_k\wedge\overline{\omega_k}\in\mathscr{A}^{1,1},$$

where $\omega_1,\cdots,\omega_g$ are orthonormal basis for the Hermitian inner product

$$\Gamma(X,\Omega^1_{X/\mathbb{C}})\times\Gamma(X,\Omega^1_{X/\mathbb{C}})\to\mathbb{C}, \quad \langle\omega,\eta\rangle\mapsto\frac{i}{2}\int_X\omega\wedge\overline{\eta}.$$

One can check that $\int_X\mu = 1$.

Definition 2.2.1 (Green Functions)**.** A **Green function (of logarithmic type) with respect to** μ is a function $g: X\times X\to\mathbb{R}$ smooth outside the diagonal $\Delta(X)\subseteq X\times X$, and satisfying the following conditions:

Fix a point $P\in X$,

- Any affine open neighbourhood U of P with local coordinate z, we have

$$g(P,z) = -\log|z-P|^2 + \text{real smooth function in } z, \quad z\in U\setminus\{P\}.$$

- For all points $z\neq P$,

$$\partial\overline{\partial}g(P,z) = -2\pi i\mu.$$

- $\int_X g(P,z)\mu = 0$.

One can prove that the Green function exists uniquely.

Remark 2.2.2. In the case of genus $g = 0$, i.e. $X = \mathbb{P}^1(\mathbb{C})$ a Riemann sphere, define a Green function on $\mathbb{P}^1(\mathbb{C})$ in terms of the affine coordinates (z,w) by

$$g(z,w) := -\log \frac{|z-w|^2}{(1+z\bar{z})(1+w\overline{w})},$$

up to an appropriate additive constant. This function with respect to the Fubini-Study form

$$\mu := \frac{i}{2\pi}\frac{dz\wedge d\bar{z}}{(1+z\bar{z})^2} \in \mathscr{A}^{1,1}.$$

There is an important formula:

Proposition 2.2.3. *Let X be a compact Riemann surface. For all smooth real-valued functions f on X,*

$$\int_X g(P,z)dd^c f + f(P) = \int_X f\mu.$$

Exercise 2.2.4. *Let $C(P,r)$ be a neighborhood of P of radius r, show that:*

- *If $g \in \mathscr{C}^\infty(C(P,r))$ and $f = \gamma \log h + \mathscr{C}^\infty$-function for some constant γ, then*

$$\lim_{r\to 0}\int_{C(P,r),\curvearrowleft} f d^c g = 0.$$

- *If $g = \log h^2 + \mathscr{C}^\infty$-function and f is continuous, then*

$$\lim_{r\to 0}\int_{C(P,r),\curvearrowleft} f d^c g = f(P).$$

Proof. (of Proposition). Write $g_P := g(P,\cdot)$, we calculate directly

$$\begin{aligned}
\int_X (g_P dd^c f - f\mu) &= \int_X (g_P dd^c f - f dd^c g_P) && (\mu = dd^c g_P)\\
&= \int_X d(g_P d^c f - f d^c g_P) && (df\wedge d^c g = dg \wedge d^c f)\\
&= \lim_{r\to 0}\int_{C(P,r),\curvearrowright} (g_P d^c f - f d^c g_P) && \text{(Stokes' formula)}\\
&= \lim_{r\to 0}\int_{C(P,r),\curvearrowleft} f d^c g_P - \lim_{r\to 0}\int_{C(P,r),\curvearrowleft} g_P d^c f && \text{(Exercise 2.2.4)}\\
&= -f(P),
\end{aligned}$$

as desired. □

Let $\omega \in \mathscr{A}^{p,q}$ $(0 \le p,q \le 1)$, define some linear operators

$$[\omega] : \mathscr{A}^{1-p,1-q} \to \mathbb{R}, \quad \eta \mapsto \int_X \omega\wedge\eta$$

and

$$\delta_P : \mathscr{C}^\infty \to \mathbb{R}, \quad f \mapsto f(P).$$

Define $dd^c[\omega](\eta) := -[\omega](dd^c\eta)$, write $g_P := g(P,\cdot)$, then the previous proposition can be expressed as the equality of operators (afterwards, they will be called currents):

$$dd^c[-g_p] + \delta_P = [\mu].$$

Exercise 2.2.5. *Let P_j ($j = 1,2$) be two different points, write $g_{P_j} := g(P_j,\cdot)$ are Green functions with respect to μ. Show*

$$g_{P_1}(P_2) - g_{P_2}(P_1) = \int_X (g_{P_1} dd^c g_{P_2} - g_{P_2} dd^c g_{P_1}) = 0.$$

It reminds us to consider the compact complex manifold $X \times X$ and the diagonal divisor $\Delta(X)$. One can choose an appropriate metric h on $\mathscr{O}_{X\times X}(\Delta(X))$ such that if s is a section of $\mathscr{O}_{X\times X}(\Delta(X))$ with $\mathrm{div}(s) = \Delta(X)$, then $-\log \|s\|_h^2$ is the Green function with respect to μ on X. This means that in the neighborhood of $\Delta(X)$ in $X \times X$, on has the expansion

$$g(z,w) = -\log|z-w|^2 + \text{real analytic function in } (z,w).$$

So we can study the analytic properties of Green functions locally in $X \times X$.

Remark 2.2.6. The Green functions can be used to define metrics on line bundles on X, under the requirements of Proposition 2.1.12 since Green functions are special Weil functions.

- We first consider the case of degree one line bundle $\mathscr{O}_X(P)$ for some prime divisor $P \in X$. Let 1_P be a meromorphic section of $\mathscr{O}_X(P)$ which is constant outside P, due to Proposition 2.1.7 one can define

$$\|1_P(z)\| := \exp\left(-\frac{1}{2}g(P,z)\right), \quad z \neq P.$$

- For the case of general line bundle

$$\mathscr{O}_X(D) = \bigotimes_P \mathscr{O}_X(P)$$

  where $D = \sum_P P$. Let 1_D be a meromorphic section of $\mathscr{O}_X(D)$ which is constant outside D, due to Remark 2.1.6 one can define

$$\|1_D(z)\| := \exp\left(-\frac{1}{2}\sum_P g(P,z)\right), \quad z \neq \mathrm{Supp}(D).$$

  We usually write $\sum_P g(P,z)$ as $g(D,z)$.

These metrics are derived from the Green function and are the metrics used in Arakelov geometry. We will emphasize them in the following sections.

Exercise 2.2.7. *Under the assumption of Remark 2.2.6, verify Proposition 2.1.12.*

2.3 Arakelov Intersection Pairing

Let K be a number field and $\mathscr{O}_K$ its ring of integers. An **arithmetic variety** X is an integral, regular, projective, flat scheme over $\mathscr{O}_K$ with generic fiber $X_K = X \times_{\mathscr{O}_K} K$ is smooth (not necessarily connected) over K. So

$$X(\mathbb{C}) := \bigsqcup_{\tau \in \mathrm{Hom}(K,\mathbb{C})} X_\tau(\mathbb{C}), \quad \text{where } X_\tau(\mathbb{C}) := \text{complex points of } X_K \times_\tau \mathbb{C},$$

is a family of compact Riemann surfaces.

An arithmetic variety with Krull dimension 2 will be called an **arithmetic surface**. The Arakelov theory of arithmetic surfaces is important because it can be calculated directly and used as important examples.

Definition 2.3.1 (Arakelov Divisors)**.** Let X be an arithmetic surface. Define the group of **Arakelov divisors** on X is the group

$$\widehat{\mathrm{Div}}(X) := \mathrm{Div}(X) \oplus \left(\bigoplus_{\tau \in \mathrm{Hom}(K,\mathbb{C})} \mathbb{R} \cdot X_\tau(\mathbb{C}) \right)^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})},$$

where $\mathrm{Div}(X)$ denotes the group of Weil divisors on X and the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on the infinite part by $\tau \mapsto \overline{\tau}$. Thus, an Arakelov divisor on X is an expression of the type $D = D_{\mathrm{fin}} + D_{\mathrm{inf}}$.

Definition 2.3.2 (Principal Arakelov Divisors)**.** Let $f \in k(X)^\times$. We associate an Arakelov divisor to f in the following way

$$\widehat{\mathrm{div}}(f) := (f)_{\mathrm{fin}} + (f)_{\mathrm{inf}},$$

where $(f)_{\mathrm{fin}}$ is the principal Weil divisor $\mathrm{div}(f)$ associated with f and

$$(f)_{\mathrm{inf}} := \sum_{\tau \in \mathrm{Hom}(K,\mathbb{C})} \left(-\frac{g_\tau(\mathrm{div}(f_\tau), z)}{2} - \log|f_\tau(z)| \right) \cdot X_\tau(\mathbb{C}),$$

where g_τ is the unique Green function on $X_\tau(\mathbb{C})$ with respect to the canonical volume form μ_τ on $X_\tau(\mathbb{C})$ invariant under $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, and f_τ is the pull-back of f by $X_\tau(\mathbb{C}) \to X$.

Exercise 2.3.3. *Check the coefficients in the sum above*

$$\gamma_\tau(f) := -\frac{g_\tau(\mathrm{div}(f_\tau), z)}{2} - \log|f_\tau(z)|$$

are constant functions in z.

These principal Arakelov divisors form a subgroup of $\widehat{\mathrm{Div}}(X)$, the quotient group is denoted by $\widehat{\mathrm{CH}}^1(X, \{\mu_\tau\})$ or simply $\widehat{\mathrm{CH}}^1(X)$, called the **arithmetic Chow group** of X.

Arakelov proved that there exists an intersection theory on an arithmetic surface:

Theorem 2.3.4 (Arakelov). *Let X be an arithmetic surface defined over $\mathscr{O}_K$, the intersection number at a closed point $x \in X$ is denoted by $i_x(\cdot,\cdot)$. With these notations, there exists a unique symmetric bilinear pairing*

$$(\cdot,\cdot) : \widehat{\mathrm{Div}}(X) \times \widehat{\mathrm{Div}}(X) \to \mathbb{R},$$

satisfying the following conditions:

- (FINITE DIVISOR, FINITE DIVISOR): $($*D, vertical divisor E lies over a finite prime* $\mathfrak{p}) =$
$$\sum_{x|\mathfrak{p}} i_x(D,E)\log\#k(x).$$
- (FINITE DIVISOR, FINITE DIVISOR): $($*horizontal divisor D, horizontal divisor* $E) = (D,E)_{\mathrm{fin}} + (D,E)_{\mathrm{inf}}$, *where*
$$(D,E)_{\mathrm{fin}} = \sum_{\mathfrak{p}\in\mathrm{Spec}(\mathscr{O}_K)} \sum_{x|\mathfrak{p}} i_x(D,E)\log\#k(x); \quad (D,E)_{\mathrm{inf}} = \sum_{\tau\in\mathrm{Hom}(K,\mathbb{C})} \frac{1}{2} g_\tau(D_\tau,E_\tau).$$
- (FINITE DIVISOR, INFINITE DIVISOR): $($*horizontal divisor D,* $X_\tau(\mathbb{C})) = \deg(D)$.
- (FINITE DIVISOR, INFINITE DIVISOR): $($*vertical divisor D,* $X_\tau(\mathbb{C})) = 0$.
- (INFINITE DIVISOR, INFINITE DIVISOR): $(X_\tau(\mathbb{C}), X_\sigma(\mathbb{C})) = 0$.
- (PRINCIPAL DIVISOR, ANY DIVISOR): $($*principal Arakelov divisor,* $\cdot) = 0$. *Therefore the pair $\widehat{\mathrm{Div}}(X) \times \widehat{\mathrm{Div}}(X) \to \mathbb{R}$ defines a symmetric bilinear form on $\widehat{\mathrm{CH}}^1(X)$.*

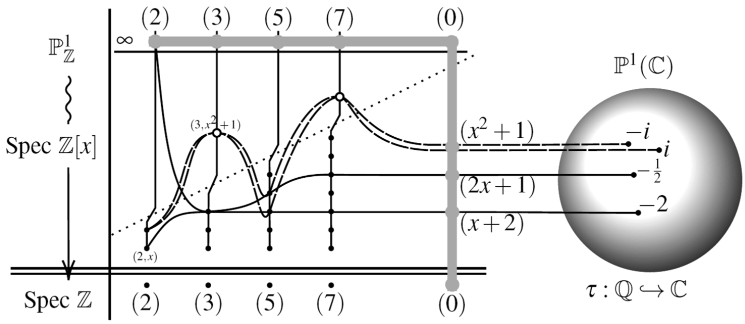
Proof. (only prove the last item). For example, given a horizontal divisor $D = D_{\mathrm{fin}}$, write $D_\tau = \sum_{i=1}^{\deg(D)} P_{i,\tau}$ such that each $P_{i,\tau}$ is prime. We have

$$\begin{aligned}
&(\widehat{\mathrm{div}}(f), D)\\
=&((f)_{\mathrm{fin}}, D) + ((f)_{\mathrm{inf}}, D)\\
=&((f)_{\mathrm{fin}}, D)_{\mathrm{fin}} + ((f)_{\mathrm{fin}}, D)_{\mathrm{inf}} + \sum_{\tau\in\mathrm{Hom}(K,\mathbb{C})} \gamma_\tau(f)\deg(D)\\
=&\sum_{\mathfrak{p}\in\mathrm{Spec}(\mathscr{O}_K)} -\log\left(\prod_{x|\mathfrak{p}} \left|f|_D\right|_x\right) + \sum_{\tau\in\mathrm{Hom}(K,\mathbb{C})} \frac{g_\tau(\mathrm{div}(f_\tau), D_\tau)}{2} + \sum_{\tau\in\mathrm{Hom}(K,\mathbb{C})} \gamma_\tau(f)\deg(D)\\
=&\sum_{\mathfrak{p}\in\mathrm{Spec}(\mathscr{O}_K)} -\log\left|\mathrm{Nm}_{k(D)/K}(f|_D)\right|_{\mathfrak{p}} + \sum_{\tau\in\mathrm{Hom}(K,\mathbb{C})} \left(\frac{g_\tau(\mathrm{div}(f_\tau), D_\tau)}{2} + \sum_{i=1}^{\deg(D)} \gamma_\tau(f)\right)\\
=&\sum_{\tau\in\mathrm{Hom}(K,\mathbb{C})} \left(\log\left|\tau\left(\mathrm{Nm}_{k(D)/K}(f|_D)\right)\right| + \sum_{i=1}^{\deg(D)} \left(\frac{g_\tau(\mathrm{div}(f_\tau), P_{i,\tau})}{2} + \gamma_\tau(f)\right)\right)\\
=&\sum_{\tau\in\mathrm{Hom}(K,\mathbb{C})} \left(\log\left|\tau\left(\prod_{\varphi\in\mathrm{Gal}(k(D)/K)} \varphi(f|_D)\right)\right| - \log\left|\tau\left(\prod_{i=1}^{\deg(D)} f_\tau(P_{i,\tau})\right)\right|\right).
\end{aligned}$$

The last term is 0 by Galois theory. □

Exercise 2.3.5. *Let $K = \mathbb{Q}, \mathscr{O}_K = \mathbb{Z}$, so there is a unique embedding $\tau : \mathbb{Q} \hookrightarrow \mathbb{C}$. Suppose $X = \mathbb{P}^1_{\mathbb{Z}}$, and the Green function g on $X_\tau(\mathbb{C})$ is given by Remark 2.2.2, let $D = \widehat{\operatorname{div}}(x^2+1)$, $E =$ the closed subscheme defined by the prime ideal $(x+2)$, show that:*

- $(D_{\text{fin}}, E)_{\text{fin}} = i_{(5,x+2)}(x^2+1, x+2)\log 5 = \log 5$.
- $(D_{\text{fin}}, E)_{\text{inf}} = \frac{1}{2}(g(i,-2) + g(-i,-2) - 2g(\infty,-2)) = \log\frac{2}{5}$.
- $\gamma_\tau(x^2+1) = -\log 2$. *Hence the intersection number $(D,E) = 0$.*



We can also identify Arakelov divisors with admissible metrized line bundles on an arithmetic surface, as in algebraic geometry. The **metrized line bundle** on an arithmetic variety X is a rank one locally free $\mathscr{O}_X$-module $\mathscr{L}$ together with a collection of non-trivial metrized line bundles $\left([X_\tau(\mathbb{C}) \to X]^*\mathscr{L}, \|\cdot\|_\tau\right)$ on compact complex manifolds $X_\tau(\mathbb{C})$, $\tau \in \operatorname{Hom}(K,\mathbb{C})$, and invariant under $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$.

Now let

$$D = D_{\text{fin}} + \sum_{\tau \in \operatorname{Hom}(K,\mathbb{C})} r_\tau \cdot X_\tau(\mathbb{C})$$

be an Arakelov divisor on an arithmetic surface. Define a metrized line bundle associated with D to be

$$\widehat{\mathscr{O}_X}(D) := \left(\mathscr{O}_X(D_{\text{fin}}), \left\{\left\|1_{D_{\text{fin}},\tau}(z)\right\|_\tau := \exp\left(-\frac{1}{2}g_\tau(D_{\text{fin},\tau}, z) - r_\tau\right)\right\}\right).$$

Because of the correspondence between Arakelov divisors and "admissible" line bundles (admissible means the line bundle has form $\widehat{\mathscr{O}_X}(D)$ when D is an Arakelov divisor), one can transplant the intersection theory of divisors to the intersection theory of line bundles. Therefore, when discussing intersections on an arithmetic surface later, we will abuse divisors and admissible line bundles. For example,

Proposition 2.3.6. *Let $D = D_{\text{fin}} + D_{\text{inf}}$ be an Arakelov divisor on an arithmetic surface X defined over $\mathscr{O}_K$, and let E be a horizontal prime divisor has form* $\operatorname{Spec}(\mathscr{O}_{k(E)})$ *with residue field $k(E)$. Then $(D,E) = \widehat{\deg}(\widehat{\mathscr{O}_X}(D)|_E)$. The right hand side can be viewed as $(\widehat{\mathscr{O}_X}(D), E)$.*

Proof. Write $D_{\mathrm{fin}} = \{(U, f)\}, D_{\mathrm{inf}} = \sum_\tau r_\tau \cdot X_\tau(\mathbb{C})$, choose a special rational section $1_{D_{\mathrm{fin}}}$, we can compute

$$\begin{aligned}
&\widehat{\deg}(\widehat{\mathscr{O}_X}(D)|_E) \\
=&\widehat{\deg}\left(\mathscr{O}_X(D_{\mathrm{fin}})|_E, \left\{\left\|1_{D_{\mathrm{fin}},\tau}(E_\tau)\right\|_\tau = \exp\left(-\frac{1}{2}g_\tau(D_{\mathrm{fin},\tau}, E_\tau) - r_\tau \deg(E)\right)\right\}\right) \\
=&\log\#\big(\mathscr{O}_X(D_{\mathrm{fin}})|_E \big/ \mathscr{O}_{k(E)}\big) - \sum_{\tau \in \mathrm{Hom}(k(E),\mathbb{C})} \log\|1_{D_{\mathrm{fin}},\tau}(E_\tau)\|_\tau \\
=&\sum_{\mathfrak{p}\in E} \mathrm{ord}_{\mathfrak{p}}(f|_E)\log\#k(\mathfrak{p}) + \sum_{\tau \in \mathrm{Hom}(k(E),\mathbb{C})}\left(\frac{1}{2}g_\tau(D_{\mathrm{fin},\tau}, E_\tau) + r_\tau \deg(E)\right) \\
=&(D_{\mathrm{fin}}, E)_{\mathrm{fin}} + (D_{\mathrm{fin}}, E)_{\mathrm{inf}} + (D_{\mathrm{inf}}, E).
\end{aligned}$$

The last term is (D, E) by Theorem 2.3.4. □

2.4 Adjunction Formula

Recall that the classicial adjunction formula in algebraic geometry states that let $f : X \to Y, g : Y \to Z$ be quasi-projective local complete intersection (l.c.i, to abbreviate) morphisms, then we have a canonical isomorphism

$$\omega_{X/Z} \cong \omega_{X/Y} \otimes_{\mathscr{O}_X} f^*\omega_{Y/Z},$$

where ω are relative canonical sheaves. This formula can be understood by differential geometry.

Let Y be a 2 dimensional compact complex manifold, and let X be a 1 dimensional regular submanifold of Y. For each $P \in X$, there are two linear spaces $T_PX \subseteq T_PY$. The complementary of T_PX in T_PY means all normal vectors of X at P relative to Y. In the language of sheaf theory, there is an exact sequence of sheaves on X

$$0 \to (\Omega^1_{X/\mathbb{C}})^\vee \to (\Omega^1_{Y/\mathbb{C}}|_X)^\vee \to \mathscr{N}\mathrm{or}_{X/Y} \to 0,$$

where $\Omega^1_{(\cdot)/\mathbb{C}}$ means the sheaf of holomorphic 1-forms, i.e. the holomorphic cotangent sheaf of $(\cdot)$, it is a locally free $\mathscr{O}_{(\cdot)}$-module with rank equal to the dimension of $(\cdot)$. Take dual and take determinant of this sequence, we get

$$\omega_{X/Y} := \bigwedge^2 \Omega^1_{Y/\mathbb{C}}|_X \cong \Omega^1_{X/\mathbb{C}} \otimes \mathscr{N}\mathrm{or}^\vee_{X/Y} = \omega_{X/\mathbb{C}} \otimes \mathscr{N}\mathrm{or}^\vee_{X/Y}.$$

If one can show $\mathscr{N}\mathrm{or}_{X/Y} \cong \mathscr{O}_Y(X)|_X$, then the adjunction formula

$$\omega_{X/\mathbb{C}} \cong \omega_{X/Y} \otimes \mathscr{O}_Y(X)|_X$$

holds and can be generalized to general cases. In algebraic geometry, one can study these sheaves locally, just use commutative algebra on each affine open subset.

Exercise 2.4.1. *Let X be a 1 dimensional regular submanifold of a 2 dimensional compact complex manifold Y. Define the conormal sheaf $\mathscr{N}\mathrm{or}^\vee_{X/Y}$ on X of $i : X \hookrightarrow Y$ to be $i^*(\mathscr{I}/\mathscr{I}^2)$, where $\mathscr{I} := \mathscr{O}_Y(-X)$ (a line bundle). Show that:*

- *$i^*(\mathscr{I}/\mathscr{I}^2) \cong i^*\mathscr{I}$, and so $\mathscr{N}\mathrm{or}^\vee_{X/Y} \cong \mathscr{O}_Y(-X)|_X$. You can check this locally: given a commutative ring A and an ideal I, there is an isomorphism $I/I^2 \otimes_A A/I \cong I \otimes_A A/I \cong I/I^2$.*

- *In particular, let $Y = X \times X$ and let $i : X \hookrightarrow X \times X$ be the diagonal embedding. Now $i^*(\mathscr{I}/\mathscr{I}^2) \cong \Omega^1_{X/\mathbb{C}}$. Locally, for a $\mathbb{C}$-algebra A, assume I is the kernel of $A \otimes_{\mathbb{C}} A \to A, a_1 \otimes a_2 \mapsto a_1a_2$ and set a A-module structure $a(a_1 \otimes a_2) := aa_1 \otimes a_2$ on it. Define*

$$\Omega^1_{A/\mathbb{C}} := \frac{\text{free } A\text{-module generated by the symbols } da, a \in A}{\langle d(a_1+a_2) - da_1 - da_2, d(a_1a_2) - a_1da_2 - a_2da_1 : a_i \in A\rangle}.$$

Recall that $\Omega^1_{X/\mathbb{C}}|_U = \widetilde{\Omega^1_{\mathscr{O}_X(U)/\mathbb{C}}}$ for any affine open subset $U \subseteq X$, so the isomorphism we want is locally given by

$$\Omega^1_{A/\mathbb{C}} \xrightarrow{\sim} I/I^2, \quad da \mapsto [a \otimes 1 - 1 \otimes a].$$

Let X be an arithmetic surface defined over $\mathscr{O}_K$. We will show that there is an analogy in Arakelov geometry.

Definition 2.4.2 (Dualizing Sheaves)**.** Let $\pi : X \to \mathscr{O}_K$ be a flat, projective, l.c.i. morphism. Let $i : X \hookrightarrow Y$ be an immersion into Y and Y is smooth over $\mathscr{O}_K$. Consider the following diagram

$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & Y \\ & \underset{\pi}{\searrow} & \downarrow p \\ & & \mathscr{O}_K \end{array}$$

Analogous to the previous discussion, we define the **relative canonical sheaf** of π to be

$$\omega_{X/\mathscr{O}_K} := \det(i^*\Omega^1_{Y/\mathscr{O}_K}) \otimes_{\mathscr{O}_X} \det(\mathscr{N}\mathrm{or}_{X/Y}),$$

where $\Omega^1_{Y/\mathscr{O}_K} := \Delta^*(\mathscr{I}/\mathscr{I}^2)$ and $\mathscr{I} := \mathscr{O}_{Y \times_{\mathscr{O}_K} Y}(-\Delta(Y)), \Delta : Y \to Y \times_{\mathscr{O}_K} Y$.

Sometimes we call $\omega_{X/\mathscr{O}_K}$ the **dualizing sheaf** with respect to π, and abbreviate it as ω_π. It can be shown that dualizing sheaf is independent of the choice of the decomposition $X \hookrightarrow Y \to \mathscr{O}_K$ up to isomorphisms.

Remark 2.4.3. Let ω_π be a dualizing sheaf with respect to π, then one can find a trace morphism $\mathrm{tr}_\pi : H^{\cdot}(X, \omega_\pi) \to \mathscr{O}_K$, such that for all coherent sheaves $\mathscr{F}$ on X, the natural pairing

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \omega_\pi) \times H^{\cdot}(X, \mathscr{F}) \longrightarrow H^{\cdot}(X, \omega_\pi) \xrightarrow{\mathrm{tr}_\pi} \mathscr{O}_K$$

followed by tr_π gives an isomorphism $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \omega_\pi) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_K}(H^{\cdot}(X, \mathscr{F}), \mathscr{O}_K)$.

Example 2.4.4. The concept of dualizing sheaves in number theory is corresponds to the codifferents. Let L/K be a finite extension of number fields and let $\theta : \mathrm{Spec}(\mathscr{O}_L) \to \mathrm{Spec}(\mathscr{O}_K)$ be the induced morphism. Now we have $\omega_\theta \cong \mathrm{Hom}_{\mathscr{O}_K}(\mathscr{O}_L, \mathscr{O}_K)$, since

$$\mathrm{Hom}_{\mathscr{O}_L}(\cdot, \mathrm{Hom}_{\mathscr{O}_K}(\mathscr{O}_L, \mathscr{O}_K)) \cong \mathrm{Hom}_{\mathscr{O}_K}(\cdot, \mathscr{O}_K) = \mathrm{Hom}_{\mathscr{O}_K}(H^0(\mathrm{Spec}(\mathscr{O}_L), \cdot), \mathscr{O}_K).$$

Recall that in algebraic number theory there is an isomorphism

$$\left\{y \in L : \mathrm{tr}_{L/K}(y\mathscr{O}_L) \subseteq \mathscr{O}_K\right\} \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_K}(\mathscr{O}_L, \mathscr{O}_K), \quad y \mapsto \mathrm{tr}_{L/K}(y\cdot).$$

So the dualizing sheaf $\omega_{\mathscr{O}_L/\mathscr{O}_K}$ in this case is just the codifferent $\mathscr{C}_{L/K}$ of a field extension.

Let us do Arakelov geometry now. Let $\tau \in \mathrm{Hom}(K,\mathbb{C})$, also write $\tau : X_\tau(\mathbb{C}) \to X$. Then by base-change, on $X_\tau(\mathbb{C})$ we have $\tau^*\omega_{X/\mathscr{O}_K} \cong \Omega^1_{X_\tau(\mathbb{C})/\mathbb{C}}$. So there is an admissible metric on $\Omega^1_{X_\tau(\mathbb{C})/\mathbb{C}} = \Delta^*\mathscr{O}_{X_\tau(\mathbb{C})\times X_\tau(\mathbb{C})}(-\Delta(X_\tau(\mathbb{C})))$, since we have the Green function on the right hand side.

Proposition 2.4.5 (Arithmetic Adjunction Formula)**.** *Let E be a horizontal prime divisor on an arithmetic surface X defined over $\mathscr{O}_K$. Suppose E has form* $\mathrm{Spec}(\mathscr{O}_{k(E)})$*, then*

$$\left(\widehat{\omega_{X/\mathscr{O}_K}} \otimes \widehat{\mathscr{O}_X}(E), E\right) = \log\#\left(\mathscr{C}_{k(E)/K}\big/\mathscr{O}_{k(E)}\right) + \frac{1}{2}\sum_{\tau\in\mathrm{Hom}(k(E),\mathbb{C})}\sum_{i\neq j} g_\tau(P_{i,\tau}, P_{j,\tau}),$$

where $E_\tau = \sum_{j=1}^{\deg(E)} P_{j,\tau}$.

Proof. We have $\omega_{E/\mathscr{O}_K} \cong \widetilde{\mathscr{C}_{k(E)/K}}$ by Example 2.4.4. On the other hand, note that $i : E \hookrightarrow X$ is a closed regular immersion, so $\omega_{E/\mathscr{O}_K} = i^*\omega_{X/\mathscr{O}_K} \otimes \mathscr{N}\mathrm{or}_{E/X}$ by classical adjunction formula. But $i^*\mathscr{O}_X(E) \cong \mathscr{N}\mathrm{or}_{E/X}$, therefore

$$\omega_{E/\mathscr{O}_K} \cong \left(\omega_{X/\mathscr{O}_K} \otimes \mathscr{O}_X(E)\right)\big|_E.$$

Now use Proposition 2.3.6 to compute

$$\left(\widehat{\omega_{X/\mathscr{O}_K}} \otimes \widehat{\mathscr{O}_X}(E), E\right) = \widehat{\deg}(\widehat{\omega_{E/\mathscr{O}_K}}) = \log\#\left(\mathscr{C}_{k(E)/K}\big/\mathscr{O}_{k(E)}\right) - \sum_{\tau\in\mathrm{Hom}(k(E),\mathbb{C})} \log\|1_\tau\|_\tau.$$

By Remark 2.2.6, the metric on $\Omega^1_{X_\tau(\mathbb{C})/\mathbb{C}} \otimes \mathscr{O}_{X_\tau(\mathbb{C})}(E_\tau)$ at $P_{j,\tau}$ is

$$\|1(P_{j,\tau})\|_\tau = \exp\left(-\frac{1}{2}\sum_{i\neq j} g_\tau(P_{i,\tau}, P_{j,\tau})\right).$$

It only needs to run out all j. □

2.5 Faltings-Riemann-Roch Theorem

Recall that the Riemann-Roch formula for a line bundle $\mathscr{L}$ on a Riemann surface X is

$$\chi(\mathscr{L}) := \dim H^0(X,\mathscr{L}) - \dim H^1(X,\mathscr{L}) = 1 - \mathrm{genus}(X) + \deg(\mathscr{L}).$$

In this section, we will introduce the analogy of this formula in Arakelov geometry.

Let V be a $\mathbb{C}$-linear space of dimension n, define $\det(V) := \bigwedge^n V$. For $\mathscr{L}$ a line bundle on a genus $g > 0$ Riemann surface X, let

$$\begin{aligned}\lambda(R\Gamma(X,\mathscr{L})) :=& \mathrm{Hom}_{\mathbb{C}}\left(\det(H^1(X,\mathscr{L})), \det(H^0(X,\mathscr{L}))\right)\\ \cong& \det(H^0(X,\mathscr{L})) \otimes \det(H^1(X,\mathscr{L}))^\vee.\end{aligned}$$

The object of this section will be to discuss volume forms on the formal difference $H^0(X,\mathscr{L}) - H^1(X,\mathscr{L})$, i.e. Hermitian inner product on $\lambda(R\Gamma(X,\mathscr{L}))$, when $\mathscr{L}$ is a metrized line bundle (we discard symbols $\|\cdot\|$ or h for simplicity) discussed in Remark 2.2.6, and discuss how these volume forms (Hermitian inner product) give rise to an Euler characteristic $\chi(\mathscr{L})$ with desirable properties, e.g. for which one has a Riemann-Roch formula.

Let D be a divisor on X, and let $P \in X$ be a point. There is an exact sequence

$$0 \to \mathscr{O}_X(D) \to \mathscr{O}_X(D+P) \to \mathbb{C}_P \to 0.$$

The metrics on $\mathscr{O}_X(D)$ and $\mathscr{O}_X(D+P)$ give rise to a metric on $\Gamma(X,\mathbb{C}_P)$, is simply the restriction of the metric on $\mathscr{O}_X(D+P)$ to the fiber at P. However, this metric is depend on D, so we write $\mathbb{C}_P$ as $\mathbb{C}_P(D)$ to emphasize this.

One has $H^1(X,\mathbb{C}_P(D)) = 0$, so there is a long exact sequence

$$0 \to \Gamma(X,\mathscr{O}_X(D)) \to \Gamma(X,\mathscr{O}_X(D+P)) \to \Gamma(X,\mathbb{C}_P(D)) \to$$
$$\to H^1(X,\mathscr{O}_X(D)) \to H^1(X,\mathscr{O}_X(D+P)) \to 0.$$

Exercise 2.5.1. *Show that:*

- *Let*
$$0 \to A_1 \to A_2 \to \cdots \to A_n \to 0$$
be a long exact sequence of finite dimensional $\mathbb{C}$-linear spaces, then
$$\left(\bigotimes_{i\geq 0} \det(A_{2i+1})\right) \otimes \left(\bigotimes_{i\geq 1} \det(A_{2i})^{\vee}\right) \cong \mathbb{C}.$$

- *The long exact sequence above gives an isomorphism*
$$\lambda(R\Gamma(X,\mathscr{O}_X(D+P))) \cong \lambda(R\Gamma(X,\mathscr{O}_X(D))) \otimes \Gamma(X,\mathbb{C}_P(D)).$$

Faltings proved the following result in 1984 (here we omit the proof):

Proposition 2.5.2 (Faltings)**.** *There is a unique way to assign to each admissible metrized line bundle $\mathscr{L}$ (admissible means the metric on $\mathscr{L}$ comes from Remark 2.2.6) on X a Hermitian inner product on $\lambda(R\Gamma(X,\mathscr{L}))$ such that the following properties hold:*

- *An isometry of metrized line bundles induces an isometry of the corresponding $\lambda(R\Gamma(X,\mathscr{L}))$.*
- *If the metric on $\mathscr{L}$ is changed by a factor $C > 0$, then the metric on $\lambda(R\Gamma(X,\mathscr{L}))$ is changed by $C^{\chi(\mathscr{L})}$.*
- *The metrics on $\lambda(R\Gamma(X,\mathscr{L}))$ are compatible with the metrics on $\mathbb{C}_P(D)$, in the following sense: Suppose D and $D+P$ are divisors on X, then the isomorphsim*
$$\lambda(R\Gamma(X,\mathscr{O}_X(D+P))) \cong \lambda(R\Gamma(X,\mathscr{O}_X(D))) \otimes \Gamma(X,\mathbb{C}_P(D))$$
is an isometry.

- *The metric on* $\lambda(R\Gamma(X,\Omega^1_{X/\mathbb{C}})) \cong \bigwedge^g(\Gamma(X,\Omega^1_{X/\mathbb{C}}))$ *is the one determined by the Hermitian inner product* $\Gamma(X,\Omega^1_{X/\mathbb{C}}) \times \Gamma(X,\Omega^1_{X/\mathbb{C}}) \to \mathbb{C}$, $\langle \omega,\eta\rangle \mapsto \frac{i}{2}\int_X \omega\wedge\overline{\eta}$.

Let us do Arakelov geometry now.

Definition 2.5.3. Let M be a finitely generated $\mathbb{Z}$-module, suppose on $M\otimes_{\mathbb{Z}}\mathbb{R}$ we have a Haar measure. Define

$$\widehat{\chi}_{\mathbb{Z}}(M) := -\log\left(\mathrm{vol}(M\otimes_{\mathbb{Z}}\mathbb{R}/M)\big/\#M_{\mathrm{tor}}\right).$$

In the case of $\mathbb{Z}$-module $\mathscr{O}_K$ when K is a number field, we choose the normalized Haar measure on $\mathscr{O}_K\otimes_{\mathbb{Z}}\mathbb{R}$ to be the one such that $\mathrm{vol}(\mathscr{O}_K\otimes_{\mathbb{Z}}\mathbb{R}/\mathscr{O}_K) = \sqrt{|\mathrm{disc}(K/\mathbb{Q})|}$.

The following definition generalizes Definition 2.5.3, since $\mathrm{disc}(\mathbb{Q}/\mathbb{Q}) = 1$.

Definition 2.5.4. Let M be a finitely generated $\mathscr{O}_K$-module, define

$$\widehat{\chi}_K(M) := \widehat{\chi}_{\mathbb{Z}}(M) - \mathrm{rank}_{\mathscr{O}_K}(M)\cdot\widehat{\chi}_{\mathbb{Z}}(\mathscr{O}_K).$$

We need a lemma to summarize some properties of bundles on an arithmetic surface.

Lemma 2.5.5. *Let X be an arithmetic surface defined over $\mathscr{O}_K$. For any coherent sheaf $\mathscr{F}$ on X, we have:*

- $H^i(X,\mathscr{F}) = 0$ *for* $i \geq 2$.
- *Denote* $i:\mathscr{O}_K \hookrightarrow K$, *then* $H^{\cdot}(X,\mathscr{F})\otimes_{\mathscr{O}_K} K \cong H^{\cdot}(X_K, i^*\mathscr{F})$. *Furthermore, if* $\tau \in \mathrm{Hom}(K,\mathbb{C})$, *then* $H^{\cdot}(X,\mathscr{F})\otimes_\tau\mathbb{C} \cong H^{\cdot}(X_\tau(\mathbb{C}),\tau^*\mathscr{F})$.

Proof. We compute the first one by using Čech cohomology. After localizing $\mathscr{O}_K$ at its primes, we are reduced to the case when R is a discrete valuation ring. Since X is projective over R, there exist homogeneous polynomials $f_1,\cdots,f_n$ with coefficients in R such that

$$X\cap[f_1=0]\cap\cdots\cap[f_n=0] \text{ is empty.}$$

Now X is covered by affine open subsets $f_j\neq 0$ for $1\leq j\leq n$, so $\check{H}^i(X,\mathscr{F}) = 0$ for $i\geq 2$. The second item is followed by flat base-change, since K and $\mathbb{C}$ are flat over $\mathscr{O}_K$. □

Let X be an arithmetic surface defined over $\mathscr{O}_K$. For an admissible metrized line bundle $\mathscr{L}$, we only need to consider $H^0(X,\mathscr{L})$ (resp. $H^0(X_\tau(\mathbb{C}),\tau^*\mathscr{L})$) and $H^1(X,\mathscr{L})$ (resp. $H^1(X_\tau(\mathbb{C}),\tau^*\mathscr{L})$) by Lemma 2.5.5.

For any embedding $\tau\in\mathrm{Hom}(K,\mathbb{C})$, we naturally have $\lambda(R\Gamma(X_\tau(\mathbb{C}),\tau^*\mathscr{L}))$, and by Proposition 2.5.2 it admits a Hermitian inner product, i.e. a volume form on the formal difference

$$H^0(X_\tau(\mathbb{C}),\tau^*\mathscr{L})) - H^1(X_\tau(\mathbb{C}),\tau^*\mathscr{L})).$$

By Lemma 2.5.5, this induces a Haar measure on

$$H^0(X,\mathscr{L})\otimes_\tau\mathbb{C} - H^1(X,\mathscr{L})\otimes_\tau\mathbb{C}.$$

This Haar measure is compatible with complex conjugation, so in fact there is a Haar measure on

$$H^0(X,\mathscr{L})\otimes_{\mathscr{O}_K}\mathbb{R} - H^1(X,\mathscr{L})\otimes_{\mathscr{O}_K}\mathbb{R}.$$

Combine these with Definition 2.5.3, we make the following definition.

Definition 2.5.6. Let $\mathscr{L}$ be an admissible metrized line bundle on an arithmetic surface X defined over $\mathscr{O}_K$, define

$$\widehat{\chi}(\mathscr{L}) := \widehat{\chi}_K(H^0(X,\mathscr{L})) - \widehat{\chi}_K(H^1(X,\mathscr{L})).$$

The main theorem is:

Theorem 2.5.7 (Faltings-Riemann-Roch)**.** *For $\mathscr{L}$ an admissible metrized line bundle on an arithmetic surface X defined over $\mathscr{O}_K$, one has the following Riemann-Roch formula*

$$\widehat{\chi}(\mathscr{L}) - \widehat{\chi}(\mathscr{O}_X) = \frac{1}{2}(\mathscr{L}, \mathscr{L} - \omega_{X/\mathscr{O}_K}),$$

where $\mathscr{O}_X$ is equipped with the standard metric.

Proof. (proof sketch). Let D be an Arakelov divisor on X such that $\mathscr{L}$ is isometric to $\widehat{\mathscr{O}_X}(D)$. Let K'/K be a finite extension, write $\phi : X' := X \times_{\mathscr{O}_K} \mathrm{Spec}(\mathscr{O}_{K'}) \to X$. We wish to show that both sides of the Riemann-Roch formula are multiplied by $[K' : K]$ if X, $\mathscr{O}_K$ and D are replaced by X', $\mathscr{O}_{K'}$ and $\phi^*(D)$. So the Riemann-Roch formula holds under base-change.

When D is the trivial divisor, there is nothing to say. By passing to a suitable extension K', we are reduced to checking that both sides of the formula change by the same amount when we add to D a divisor of the following kinds:

- A real multiple of an infinite fibre of X.
- An irreducible component of a fibre of $X \to \mathscr{O}_K$ at a closed point of $\mathrm{Spec}(\mathscr{O}_K)$.
- The image of a section $s : \mathrm{Spec}(\mathscr{O}_K) \to X$.

Some of them are deduced from the arithmetic adjunction formula. □

Chapter 3

Higher Arakelov Geometry

In this chapter, we introduce the arithmetic intersection theory on arithmetic varieties developed by Gillet and Soulé.

3.1 Some Intersection Theory and K-Theory

There is an important model establishes the intersection theory locally:

Remark 3.1.1. Let R be a noetherian regular local ring with residue field k. A finitely generated R-module has finite length if and only if it is supported at the closed point of $\mathrm{Spec}(R)$. By dévissage, the K_0 of the category of modules of finite length is isomorphic to the K_0 of the category of k-linear spaces, i.e. to $\mathbb{Z}$. Now let M,N are finitely generated R-modules (hence have finite length), the supports of which intersect only at the closed point of $\mathrm{Spec}(R)$. Serre defines their **intersection multiplicity**

$$i(M,N) := \sum_{k\geq 0}(-1)^k \ell(\mathrm{Tor}_k^R(M,N)).$$

This formula will be served as the standard model for the general intersection theory.

Let X be a noetherian, regular, separated scheme of dimension d defined over a noetherian ring.

For any $p \in \mathbb{Z}_{\geq 0}$, denoted by $X^{(p)}$ the set of points of codimension p in X. Let $Z^p(X)$ be the free abelian group generated by $X^{(p)}$, the elements in it are called **p-cycles**. Two p-cycles Z_1, Z_2 are called **rationally equivalent** if there exist finitely many functions $f_i \in k(y_i)^\times$, $y_i \in X^{(p-1)}$ such that

$$Z_2 - Z_1 = \sum_i \mathrm{div}(f_i),$$

where

$$\mathrm{div}(f_i) = \sum_{x\in X^{(p)}\cap\overline{\{y_i\}}} \mathrm{ord}_{\mathscr{O}_{\overline{\{y_i\}},x}}(f_i)\cdot\overline{\{x\}}.$$

Definition 3.1.2 (Chow Groups)**.** The p-th **Chow group** $\mathrm{CH}^p(X)$ of X is the quotient group

$$\mathrm{CH}^p(X) := Z^p(X)/\text{rational equivalence}.$$

For a closed subscheme $Y \subseteq X$ we define $Z^p_Y(X)$ as the group of cycles of codimension p on X supported in the closed subset attached to Y, then define

$$\mathrm{CH}^p_Y(X) := Z^p_Y(X)/\langle \mathrm{div}(f) : f \in k(y)^\times, y \in X^{(p-1)} \cap Y \rangle,$$

call it the Chow group of codimension p of X with supports in Y.

Definition 3.1.3 (Intersections)**.** Two cycles $Y \in Z^p(X), Z \in Z^q(X)$ **intersect properly**, if $\mathrm{codim}_X(Y \cap Z) = p+q$. Assume Y, Z intersect properly, define the **intersection multiplicity** $i_x(Y,Z)$ for $x \in Y \cap Z \cap X^{(p+q)}$ is the integer

$$i_x(Y,Z) := \sum_{k \geq 0} (-1)^k \ell_{\mathscr{O}_{X,x}}(\mathrm{Tor}_k^{\mathscr{O}_{X,x}}(\mathscr{O}_{Y,x}, \mathscr{O}_{Z,x})).$$

Write $(\cdot)_{\mathbb{Q}} := (\cdot) \otimes_{\mathbb{Z}} \mathbb{Q}$ (the reason for tensor $\mathbb{Q}$ is given by K-theory). The main conclusions of this section are as follows.

Theorem 3.1.4. *Let Y, Z be closed subschemes of X, then there exists a bilinear pairing*

$$(\cdot,\cdot) : \mathrm{CH}^p_Y(X)_{\mathbb{Q}} \times \mathrm{CH}^q_Z(X)_{\mathbb{Q}} \to \mathrm{CH}^{p+q}_{Y\cap Z}(X)_{\mathbb{Q}}$$

satisfying the following properties:

- $\bigoplus_Y \bigoplus_p \mathrm{CH}^p_Y(X)_{\mathbb{Q}}$ *is a commutative ring with unit* $[X] \in \mathrm{CH}^0(X)$.
- *It is compatible with change of supports* $\mathrm{CH}^p_Y(X)_{\mathbb{Q}} \to \mathrm{CH}^p_{Y'}(X)_{\mathbb{Q}}$ *associated to inclusions* $Y \subseteq Y'$.
- *For* $[Y_1] \in \mathrm{CH}^p_Y(X), [Z_1] \in \mathrm{CH}^p_Z(X)$ *with* Y_1, Z_1 *intersect properly, we have*

$$([Y_1],[Z_1]) \mapsto \left[\sum_{x \in Y_1 \cap Z_1 \cap X^{(p+q)}} i_x(Y_1, Z_1) \cdot \overline{\{x\}} \right].$$

In particular, there exists a unique pairing

$$\mathrm{CH}^p(X)_{\mathbb{Q}} \otimes \mathrm{CH}^q(X)_{\mathbb{Q}} \to \mathrm{CH}^{p+q}(X)_{\mathbb{Q}}$$

such that for $Y \in Z^p(X), Z \in Z^q(X)$ interesting properly, we have

$$([Y],[Z]) \mapsto \left[\sum_{x \in Y \cap Z \cap X^{(p+q)}} i_x(Y, Z) \cdot \overline{\{x\}} \right].$$

The pairing above is given by tensor product of bundles in K_0 group. So if we want to prove this theorem, we need to introduce some K-theory.

Definition 3.1.5 (Grothendieck Groups)**.** Let Y be a closed subscheme of X. Define:

- $K_0(X)$ to be the **Grothendieck group** of coherent locally free $\mathscr{O}_X$-modules (i.e. finite dimensional vector bundles). More precisely,

$$K_0(X) := \frac{\text{the free abelian group generated by coherent locally free } \mathscr{O}_X\text{-modules}}{\langle \mathscr{F}' - \mathscr{F} + \mathscr{F}'' : 0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0 \text{ exact} \rangle}.$$

- $K_0'(X)$ to be the Grothendieck group of coherent $\mathscr{O}_X$-modules.
- $K_0^Y(X)$ to be the Grothendieck group of bounded complexes of locally free $\mathscr{O}_X$-modules acyclic outside Y modulo quasi-isomorphisms and $(\mathfrak{F}.' - \mathfrak{F}. + \mathfrak{F}.'')$ if there is an exact sequence $0 \to \mathfrak{F}.' \to \mathfrak{F}. \to \mathfrak{F}.'' \to 0$.

Since our X is regular, $K_0(X) \cong K_0'(X)$. The map from $K_0'(X)$ to $K_0(X)$ is given by the finite and locally free resolution

$$0 \to \mathscr{F}_n \to \cdots \to \mathscr{F}_1 \to \mathscr{F}_0 \to \mathscr{F} \to 0$$

of a coherent $\mathscr{O}_X$-module $\mathscr{F}$, for this we can send $[\mathscr{F}]$ to $\sum_{i=0}^n (-1)^i [\mathscr{F}_i]$.

We now review some facts in algebraic geometry, but omit the proof.

Proposition 3.1.6. *Let Y, Z (not necessarily regular) be closed subschemes of X, their closed immersions to X are denoted as i.*

- *(Excision Theorem). There is an exact sequence $K_0'(Y) \to K_0'(X) \to K_0'(X \setminus Y) \to 0$.*
- *There is a bilinear pairing*

$$K_0^Y(X) \times K_0^Z(X) \to K_0^{Y \cap Z}(X), \quad ([\mathfrak{F}.], [\mathfrak{G}.]) \mapsto [\mathrm{Tot}(\mathfrak{F}. \otimes \mathfrak{G}.)],$$

where the total complex of a double complex $\mathfrak{F}. \otimes \mathfrak{G}.$, $\mathfrak{F}. = \{\mathscr{F}_i, d_i\}, \mathfrak{G}. = \{\mathscr{G}_j, \delta_j\}$, is defined by

$$\mathrm{Tot}(\mathfrak{F}. \otimes \mathfrak{G}.) := \left\{ \left(\bigoplus_{i+j=n} \mathscr{F}_i \otimes \mathscr{G}_j \right)_n, \bigoplus_{i+j=n} \Big(d_i \otimes \mathrm{id} + (-1)^i \mathrm{id} \otimes \delta_j \Big) \right\}.$$

- *There is an isomorphism $K_0'(Y) \xrightarrow{\sim} K_0^Y(X), [\mathscr{F}] \mapsto [\mathfrak{F}.]$, where $\mathfrak{F}.$ is a finite free resolution of $i_* \mathscr{F}$.*
- *(Projection Formula). Let $f : X \to X'$ be a proper morphism. The homomorphism $f^* : K_0(X') \to K_0(X), [\mathscr{F}'] \mapsto [f^* \mathscr{F}']$ and the homomorphism $f_* : K_0'(X) \to K_0'(X'), [\mathscr{F}] \mapsto \sum_i (-1)^i [R^i f_* \mathscr{F}]$ satisfy the formula*

$$f_*(f^*[\mathscr{F}'] \otimes [\mathscr{F}]) = [\mathscr{F}'] \otimes f_*[\mathscr{F}], \quad \text{for } [\mathscr{F}] \in K_0'(X), [\mathscr{F}'] \in K_0(X').$$

In order to state Theorem 3.1.4 using K-theory, we make the following definition.

Definition 3.1.7. On $K_0^Y(X)$ we define a decreasing filtration

$$K_0^Y(X) = F^0 K_0^Y(X) \supseteq F^1 K_0^Y(X) \supseteq \cdots \supseteq F^d K_0^Y(X) \supseteq F^{d+1} K_0^Y(X) = \{0\}$$

by

$$F^p K_0^Y(X) := \bigcup_{Z \subseteq Y, \mathrm{codim}_X Z \geq p} \mathrm{im}\Big(K_0^Z(X) \to K_0^Y(X) \Big).$$

Define $\mathrm{Gr}^p K_0^Y(X) := F^p K_0^Y(X) / F^{p+1} K_0^Y(X)$.

Theorem 3.1.8. *Using the terminologies above,*

- $F^pK_0^Y(X)_{\mathbb{Q}} \cdot F^qK_0^Z(X)_{\mathbb{Q}} \subseteq F^{p+q}K_0^{Y\cap Z}(X)_{\mathbb{Q}}$, *given by take the total complex of the tensor product double complex in Proposition 3.1.6.*
- *Let* $Z \in Z_Y^p(X)$ *be an irreducible cycle, then we can take a finite locally free resolution of* $i_*\mathscr{O}_Z$. *This induces an isomorphism* $\mathrm{CH}_Y^p(X)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Gr}^pK_0^Y(X)_{\mathbb{Q}}$.

Proof. (of Theorem 3.1.4). It is not hard to see that Theorem 3.1.8 implies Theorem 3.1.4, since on $\mathrm{Gr}^{\cdot}K_0(X)_{\mathbb{Q}}$ we can do tensors naturally. The intersection number $i_x(\cdot,\cdot)$ comes from Proposition 3.1.6, because we can define a bilinear pairing

$$K_0'(Y)\times K_0'(Z)\to K_0'(Y\cap Z),\quad ([\mathscr{F}],[\mathscr{G}])\mapsto \sum_k(-1)^k[\mathscr{H}_k(\mathrm{Tot}(\mathfrak{P}.\otimes\mathfrak{Q}.))],$$

where $\mathfrak{P}.\to i_*\mathscr{F}\to 0$, $\mathfrak{Q}.\to i_*\mathscr{G}\to 0$ are free resolutions of $\mathscr{O}_X$-modules (here i denotes the closed immersion to X). By homological algebra we have $\mathscr{H}_k(\mathrm{Tot}(\mathfrak{P}.\otimes\mathfrak{Q}.))\cong \mathscr{T}\mathrm{or}_k^{\mathscr{O}_X}(i_*\mathscr{F},i_*\mathscr{G})$, so the intersection number $i_x(Y,Z)$ will be defined to be the local information at $x\in Y\cap Z$ of the image of $([\mathscr{O}_Y],[\mathscr{O}_Z])$. Since $\mathscr{T}\mathrm{or}$ and i_* commute with colimits, this is the alternating sum of the lengthes of the stalks at x of Tor sheaves, due to Remark 3.1.1. □

Note that the complex

$$\cdots\to 0\to 0\to i_*\mathscr{F}\otimes i_*\mathscr{G}\to 0\to 0\to\cdots$$

is isomorphic to $i_*\mathscr{F}\otimes\mathfrak{Q}.$ or $\mathfrak{P}.\otimes i_*\mathscr{G}$ in the derived category, so the right hand side of the map above is equal to $[i_*\mathscr{F}\otimes i_*\mathscr{G}]$ in $K_0'(X)$. Therefore the intersection bilinear form is "almost" the tensor product of $\mathscr{O}_X$-modules.

Remark 3.1.9. When X is a smooth variety over a field, the Chow groups can also be defined by using sheaf cohomology. The **Bloch's formula** tells us that there is an isomorphism:

$$\mathrm{CH}^p(X)\cong H^p\Big(X,\big(U\mapsto \pi_{p+1}\mathscr{B}\mathscr{Q}\{\text{finitely generated projective }\mathscr{O}_X(U)\text{-module}\}\big)^{\dagger}\Big),$$

where † means sheafification, $\mathscr{Q}$ means add some arrows to the category, $\mathscr{B}$ means geometric realization (to make a category into a topological space) and π_{p+1} means the $(p+1)$-th homotopy group. Some technical tools can be found in Quillen's higher K-theory and homotopy theory.

We will give a proof sketch of Theorem 3.1.8. Some preparation is needed.

Definition 3.1.10 (λ-Rings)**.** A **λ-ring** is a unitary ring R with operations $\lambda^k: R\to R$ ($k\geq 0$), satisfying

- $\lambda^0=1$; $\lambda^1=\mathrm{id}$; $\lambda^k(1)=0(\forall k>1)$.
- $\lambda^k(x+y)=\sum_{i=0}^k\lambda^i(x)\cdot\lambda^{k-i}(y)$.
- $\lambda^k(xy)=P_k(\lambda^1(x),\cdots,\lambda^k(x);\lambda^1(y),\cdots,\lambda^k(y))$, where P_k is a integral coefficient polynomial in $2k$ variables s.t. $P_k(e_1,\cdots,e_k;f_1,\cdots,f_k)$ is the coefficient of t^k in the expression $\prod_{i=1}^k\prod_{j=1}^k(1+tx_iy_j)$, where $e_1,\cdots,e_k$ (resp. $f_1,\cdots,f_k$) are elementary symmetric polynomials in $x_1,\cdots,x_k$ (resp. $y_1,\cdots,y_k$).

- $\lambda^k(\lambda^l(x)) = Q_{k,l}(\lambda^1(x), \cdots, \lambda^{kl}(x))$, where $Q_{k,l}$ is a integral coefficient polynomial in kl variables s.t. $Q_{k,l}(e_1, \cdots, e_{kl})$ is the coefficient of t^k in the expression $\prod_{1\le i_1<i_2<\cdots<i_k\le kl}(1+tx_{i_1}x_{i_2}\cdots x_{i_k})$, where $e_1, \cdots, e_{kl}$ are elementary symmetric polynomials in $x_1, \cdots, x_{kl}$.

Note that P_k, Q_{kl} are not depend on R.

Exercise 3.1.11. *$P_1(x;y) = xy$; $P_2(x,y;z,w) = x^2w + z^2y - 2yw$.*

The concept of λ-rings comes from the analogy of operations on vector bundles. Indeed, we can roughly view $(K_0, +, \otimes, \wedge)$ as a λ-ring with unit is the trivial bundle $[\mathscr{O}]$.

Exercise 3.1.12. *If on $X = \mathbb{P}^1(\mathbb{C})$ there is*

$$\bigwedge^k \left(\mathscr{O}_X(l)^{\oplus n}\right) = \mathscr{O}_X(kl)^{\oplus \Phi(n,k)},$$

find $\Phi(n,k)$. In particular, $\bigwedge^n(\mathscr{O}_X(l)^{\oplus n}) = \mathscr{O}_X(nl)$.

Definition 3.1.13 (Adams Operators)**.** Write $\lambda_t(x) := \sum_{k\ge 0}\lambda^k(x)t^k$. Put

$$\psi_{-t}(x) := -\frac{t}{\lambda_t(x)} \cdot \frac{d\lambda_t(x)}{dt},$$

and $\psi_t(x) := \sum_{k\ge 1}\psi^k(x)t^k$. The operators $\psi^k : R \to R$ are called the **Adams operators** on the λ-ring R.

There is an important principle in algebraic topology, called the **splitting principle**. That is, to check universal relations among operations on λ-rings, it is sufficient to check these on elements of the form $x = x_1 + \cdots + x_n$ with $\lambda^k(x_i) = 0$ for all $k > 1, i = 1, \cdots, n$. This is because for a vector bundle E on X, there is a **tautological exact sequence** on $\mathbb{P}E$

$$0 \to \mathscr{O}_{\mathbb{P}E}(-1) \to \pi^*E \to \text{Quot} \to 0,$$

where $\pi : \mathbb{P}E \to X$, whose fiber at a point $x \in X$ is the usual projective space of lines in the fiber E_x. The fact is that $[\pi^*E]$ completely determines $[E]$ on X and $\text{rank}(\text{Quot}) < \text{rank}(E)$, so one can continue this process by induction. Finally we get some line bundles from E. Since in the K_0 group we modulo exact sequences, the splitting principle is reasonable.

Proposition 3.1.14. *Let ψ^k ($k \ge 1$) be Adams operators on a λ-ring R.*

- *ψ^k are ring endomorphisms.*
- *$\psi^k \circ \psi^l = \psi^{kl} = \psi^l \circ \psi^k$.*
- *$\psi^k = \text{New}_k(\lambda^1, \cdots, \lambda^k)$, where New_k is the k-th Newton polynomial.*

Proof. We only prove the first one. Obviously, $\lambda_t(x+y) = \lambda_t(x)\cdot\lambda_t(y)$, so ψ^k preserves addition. To check ψ^k preserves multiplication, we use splitting principle. Let $x, y \in R$ with $\lambda^k(x) = \lambda^k(y) = 0$ for all $k > 1$, hence $\lambda^k(xy) = 0$ for all $k > 1$. Then $\lambda_t(xy) = 1 + txy$ and therefore $\psi_{-t}(xy) = \frac{-txy}{1+txy}$. This implies $\psi_t(xy) = \sum_{k\ge 1}(txy)^k$, so $\psi^k(xy) = (xy)^k = \psi^k(x)\psi^k(y)$. $\square$

Exercise 3.1.15. *There is a unique λ-ring structure on $\mathbb{Z}$, given by*

$$\lambda^k : \mathbb{Z} \to \mathbb{Z}, \quad n \mapsto \binom{n}{k} = \dim\left(\bigwedge^k \mathbb{Q}^{\oplus n}\right).$$

The Adams operators on this λ-ring are $\psi^k = \mathrm{id}$ for all $k \geq 1$. Check this.

Proof. (proof sketch of Theorem 3.1.8). We take the following steps.

- (Fact i). There exists a functorial λ-ring structure on $\bigoplus_{Y \subseteq X} K_0^Y(X)$ such that λ^k maps $K_0^Y(X)$ to itself for every $k \geq 0$. Note that this does not mean that $K_0^Y(X)$ has an induced λ-ring structure, in fact it may not have a unit. In particular, if $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(R/(a))$, then

$$\psi^k : K_0^Y(X) \to K_0^Y(X), \quad [\Sigma_a] := [\cdots \to 0 \to R \xrightarrow{a} R \to 0 \to \cdots] \mapsto k \cdot [\Sigma_a].$$

- (Fact ii). Let A be a commutative ring, write $K_\cdot(A)$ instead of $K_\cdot(\mathrm{Spec}(A))$. In particular, if $A = F$ is a field, then

$$\begin{aligned} &K_0(F) \cong \mathbb{Z}, \\ &K_1(F) \cong F^\times, \\ &K_2(F) \cong F^\times \otimes_{\mathbb{Z}} F^\times \big/ \langle x \otimes (1-x) : x \in F^\times \setminus \{1\} \rangle. \end{aligned}$$

  The Adams operators ψ^k act on $K_0(F)$ by id, on $K_1(F)$ by multiplication by k. Here, the higher K-theory appears, but beyond this proof, we will not use it.

- (Claim iii). Let $Y \subseteq X$ be a closed subscheme with codimension p, then there exists an exact sequence

$$0 \longrightarrow F^{p+1} K_0^Y(X) \longrightarrow K_0^Y(X) \longrightarrow \bigoplus_{V \in Y \cap X^{(p)} \text{ closed}} K_0^V(\mathscr{O}_{X,V}) \longrightarrow 0.$$

  Proof. Since X is regular, by Proposition 3.1.6 we have for $Z \subseteq Y$ a closed subscheme, there is an exact sequence

$$0 \longrightarrow \mathrm{im}\big(K_0^Z(X) \to K_0^Y(X)\big) \longrightarrow K_0^Y(X) \longrightarrow K_0^{Y \setminus Z}(X \setminus Z) \longrightarrow 0.$$

  Take colimit over all closed subschemes $Z \subseteq Y$ with codimension $\geq p+1$, we get

$$0 \longrightarrow F^{p+1} K_0^Y(X) \longrightarrow K_0^Y(X) \longrightarrow \varinjlim_{Z \subseteq Y, \mathrm{codim}_X Z \geq p+1} K_0^{Y \setminus Z}(X \setminus Z) \longrightarrow 0.$$

  The colimit in the sequence above is $\displaystyle\bigoplus_{V \in Y \cap X^{(p)} \text{ closed}} K_0^V(\mathscr{O}_{X,V})$. □

- (Fact iv). Fix $k \geq 2$. For $i \geq 0$, denote $K_0^Y(X)^{[i]} := \{x \in K_0^Y(X)_{\mathbb{Q}} : \psi^k(x) = k^i x\}$ (it does not depend on the choice of k by Proposition 3.1.14, but this fact is not trivial). Then we have (i.e. the action of ψ^k is diagonalizable with eigenvalues in $\{k^0, \cdots, k^d\}$)

$$F^p K_0^Y(X)_{\mathbb{Q}} \cong \bigoplus_{i \geq p} K_0^Y(X)^{[i]}.$$

This was first proven by Grothendieck. The idea of proof is to use descending induction on $\operatorname{codim}_X(Y)$. To start with, let Y be a closed point, then $K_0^Y(X) \cong K_0'(Y) \cong K_0(k(Y)) \cong \mathbb{Z}$ (this is not a λ-ring, otherwise, it will contradict Exercise 3.1.15), so we only need to check the action of ψ^k on some element is multiplication by k^d. This is given by (Fact i), since one can compute $\psi^k\big(\big[\bigotimes_{i=1}^d \Sigma_{a_i}\big]\big) = k^d\big[\bigotimes_{i=1}^d \Sigma_{a_i}\big]$ for some a_i. To prove the general case, one should use (Claim iii).

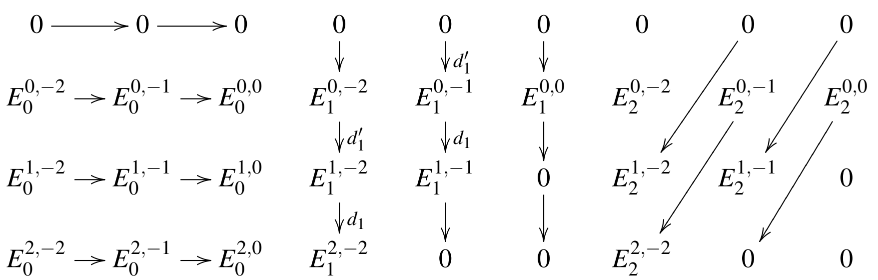
- (Claim v). $F^p K_0^Y(X)_{\mathbb{Q}} \cdot F^q K_0^Z(X)_{\mathbb{Q}} \subseteq F^{p+q} K_0^{Y \cap Z}(X)_{\mathbb{Q}}$.

Proof. Let $x \in F^p K_0^Y(X)_{\mathbb{Q}}$ and $y \in F^q K_0^Z(X)_{\mathbb{Q}}$, by (Fact iv) we have $x = \sum_{i \geq p} x_i$ with $x_i \in K_0^Y(X)^{[i]}$ and $y = \sum_{j \geq q} y_j$ with $y_j \in K_0^Z(X)^{[j]}$. Therefore $xy = \sum_{i,j} x_i y_j$ with $\psi^k(x_i y_j) = k^{i+j} x_i y_j$ for $i + j \geq p + q$. So $xy \in F^{p+q} K_0^{Y \cap Z}(X)_{\mathbb{Q}}$. □

- (Fact vi). The higher K groups fit a spectral sequence, its E_1 page is

$$E_{1,Y}^{p,q}(X) = \begin{cases} \bigoplus_{x \in Y \cap X^{(p)}} K_{-p-q}(k(x)), & p \geq 0, p+q \leq 0, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, (Fact ii) implies $E_{1,Y}^{p-1,-p}(X) \cong \bigoplus_{y \in Y \cap X^{(p-1)}} k(y)^\times$ and $E_{1,Y}^{p,-p}(X) \cong \bigoplus_{x \in Y \cap X^{(p)}} \mathbb{Z} \cong Z_Y^p(X)$.



The maps d_1 in the E_1 page are

$$d_1 : \bigoplus_{y \in Y \cap X^{(p-1)}} k(y)^\times \to Z_Y^p(X), \quad (f_y) \mapsto \sum_y \operatorname{div}(f_y).$$

It is easy to see $Z_Y^p(X)/\operatorname{im}(d_1) \cong \mathrm{CH}_Y^p(X)$, i.e. $E_{2,Y}^{p,-p}(X) \cong \mathrm{CH}_Y^p(X)$. Similarly,

$$E_{2,Y}^{p-1,-p}(X) \cong \left\{ (f_y) \in \bigoplus_{y \in Y \cap X^{(p-1)}} k(y)^\times : \sum_y \operatorname{div}(f_y) = 0 \right\} \Big/ \operatorname{im}(d_1').$$

- (Fact vii). There are λ-ring structures $\lambda^k : E_{r,Y}^{p,q}(X) \to E_{r,Y}^{p,q}(X)$, and they converge to $\lambda^k : K_{-p-q}^Y(X) \to K_{-p-q}^Y(X)$. For the isomorphisms in (Fact vi)

$$\varepsilon : \bigoplus_{x \in Y \cap X^{(p)}} K_{-p-q}(k(x)) \xrightarrow{\sim} E_{1,Y}^{p,q}(X),$$

the Adams operators ψ^k satisfy $\psi^k \circ \varepsilon = k^p \cdot \varepsilon \circ \psi^k$.

This is proven by using Quillen's higher K-theory to compute cohomology groups.

- (Claim viii). $E_{2,Y}^{p,-p}(X)_{\mathbb{Q}} \cong \mathrm{Gr}^p K_0^Y(X)_{\mathbb{Q}}$.

Proof. (Fact vii) implies that ψ^k acts on $E_{r,Y}^{p-1,-p}(X)$ by multiplication by k^p, on $E_{r,Y}^{p-1+r,-(p-1+r)}(X)$ by multiplication by k^{p-1+r}. Since the differentials

$$d_r^{p-1,-p} : E_{r,Y}^{p-1,-p}(X) \to E_{r,Y}^{p-1+r,-(p-1+r)}$$

commute with ψ^k, we have $k^p(k^{r-1}-1)d_r^{p-1,-p} = 0$. Hence, if $r \geq 2$, after tensoring with $\mathbb{Q}$, $d_r^{p-1,-p}$ vanishes, i.e. $E_{2,Y}^{p,-p}(X)_{\mathbb{Q}} \cong E_{\infty,Y}^{p,-p}(X)_{\mathbb{Q}}$. Now we can compute the E_∞ page by filtering, and finally get $E_{\infty,Y}^{p,-p}(X)_{\mathbb{Q}} = \mathrm{Gr}^p K_0^Y(X)_{\mathbb{Q}}$. □

To summarize, we obtain $\mathrm{CH}_Y^p(X)_{\mathbb{Q}} \cong \mathrm{Gr}^p K_0^Y(X)_{\mathbb{Q}}$. □

Exercise 3.1.16. *As a computable case, we compute the Chow groups of $X = \mathbb{P}^1(\mathbb{C})$.*

- *There is a split exact sequence*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} K_0'(X) \xrightarrow{\beta} \mathrm{Pic}(X) \longrightarrow 0,$$

where $\alpha : n \mapsto n[\mathscr{O}_X]$; $\alpha^{-1} = \mathrm{rank}$; $\beta = \det$; $\beta^{-1} : \mathscr{O}_X(P) \mapsto [\mathbb{C}_P] = [\mathscr{O}_X(P)] - [\mathscr{O}_X]$.

- *The exact sequence above induces an isomorphism of groups*

$$K_0'(X) \xrightarrow{\sim} \mathrm{Pic}(X) \oplus \mathbb{Z} \xrightarrow{\sim} \mathbb{Z}^2, \quad [\mathscr{F}] \mapsto (\det(\mathscr{F}), \mathrm{rank}(\mathscr{F})) \mapsto (\deg(\det(\mathscr{F})), \mathrm{rank}(\mathscr{F})).$$

Moreover, if one makes $\mathbb{Z}^2$ into a ring by define $(a,b)\cdot(c,d) := (ad+bc, bd)$, so $\mathbb{Z}^2 \cong \mathbb{Z}[x]/(x^2)$. This makes $K_0'(X)$ into a ring, the multiplicative structure is given by the tensor product of $\mathscr{O}_X$-modules. (Indeed, $\mathbb{Z}[x]/(x^2)$ is the Chow ring $\mathrm{CH}^\cdot(X)$ of X, where $x = [P]$ corresponds to the skyscraper sheaf $[\mathbb{C}_P]$ in $K_0'(X)$).

- *Verify $\mathrm{Gr}^0 K_0^X(X) \cong \mathbb{Z}$, $\mathrm{Gr}^1 K_0^X(X) \cong \mathbb{Z}$. Hence the Chow ring $\mathrm{CH}^\cdot(X) = \mathbb{Z} \oplus \mathbb{Z}$ by Theorem 3.1.8.*

- *Let P be a closed point in X. Verify $\mathrm{Gr}^0 K_0^P(X) = 0$, $\mathrm{Gr}^1 K_0^P(X) \cong \mathbb{Z}$. Hence the Chow ring with supports in P is $\mathrm{CH}_P^\cdot(X) = \mathbb{Z}$ by Theorem 3.1.8.*

- *Consider the natural λ-ring structure given by the wedge product on $K_0'(X) \cong \mathbb{Z}^2$. Verify the second component of λ^2 is $(m,n) \mapsto \frac{n(n-1)}{2}$, and the second component of ψ^2 is $(m,n) \mapsto n$.*

3.2 Green Currents

In this section, we introduce some preliminaries of complex geometry. The arithmetic variety must have a smooth generic fiber, so the infinite part is a smooth projective complex variety. Now let X be a smooth projective complex manifold, we will define some currents on it with respect to some closed irreducible subvarieties $Z \subseteq X$. But Z may not be smooth! Therefore, in order to make a definition of integrating on Z, one may need the resolution of singularities.

Now we review some basic concepts of differential forms.

Let X be a smooth projective complex equidimensional variety of complex dimension d, denote

$$\mathscr{A}^{p,q} := \text{the linear space of } \mathbb{C}\text{-valued differential forms of type } (p,q).$$

More precisely, if $(z_1,\cdots,z_d)$ are local coordinates, then any element in $\mathscr{A}^{p,q}$ has form

$$\sum_{\substack{1\le i_1<\cdots<i_p\le d\\ 1\le j_1<\cdots<j_q\le d}}^{<\infty} f_{i_1,\cdots,i_p;j_1,\cdots,j_q}(z_1,\cdots,z_d;\overline{z_1},\cdots,\overline{z_d})dz_{i_1}\wedge\cdots\wedge dz_{i_p}\wedge d\overline{z_{j_1}}\wedge\cdots\wedge d\overline{z_{j_q}},$$

where $f_{i_1,\cdots,i_p;j_1,\cdots,j_q}$ are smooth functions. Denote by $\mathscr{A}^n := \bigoplus_{p+q=n}\mathscr{A}^{p,q}$ the space of differential forms of degree n, and denote by ∂, $\overline{\partial}$, d the usual differentials.

Definition 3.2.1 (Currents)**.** Define

$$\mathscr{D}_n := \left\{ F:\mathscr{A}^n\to\mathbb{C} \text{ linear}: \begin{array}{l}\text{for any compact } K, \text{ if } \{\omega_r\}\subseteq\mathscr{A}^n \text{ s.t. } \mathrm{Supp}(\omega_r)\subseteq K\\ \text{and all derivatives of all coefficients of } \omega_r \rightrightarrows 0 \text{ on } K\\ \text{if } r\to\infty, \text{ then } F(\omega_r)\to 0.\end{array}\right\},$$

then we obtain the decomposition $\mathscr{D}_n = \bigoplus_{p+q=n}\mathscr{D}_{p,q}$. Now define the space of n-**currents** (resp. (p,q)-currents) to be $\mathscr{D}^n := \mathscr{D}_{d-n}$ (resp. $\mathscr{D}^{p,q} := \mathscr{D}_{d-p,d-q}$).

The differentials ∂, $\overline{\partial}$, d induce:

- $\partial:\mathscr{D}^{p,q}\longrightarrow\mathscr{D}^{p+1,q}$, given by

$$\big(T:\mathscr{A}^{d-p,d-q}\to\mathbb{C},\omega\mapsto T(\omega)\big)\longmapsto\big(\partial T:\mathscr{A}^{d-p-1,d-q}\to\mathbb{C},\omega\mapsto(-1)^{p+q+1}T(\partial\omega)\big).$$

- $\overline{\partial}:\mathscr{D}^{p,q}\longrightarrow\mathscr{D}^{p,q+1}$, given by

$$\big(T:\mathscr{A}^{d-p,d-q}\to\mathbb{C},\omega\mapsto T(\omega)\big)\longmapsto\big(\overline{\partial} T:\mathscr{A}^{d-p,d-q-1}\to\mathbb{C},\omega\mapsto(-1)^{p+q+1}T(\overline{\partial}\omega)\big).$$

- $d=\partial+\overline{\partial}:\mathscr{D}^{p,q}\longrightarrow\mathscr{D}^{p+q+1}$, given by

$$\big(T:\mathscr{A}^{d-p,d-q}\to\mathbb{C},\omega\mapsto T(\omega)\big)\longmapsto\big(dT:\mathscr{A}^{d-p-q-1}\to\mathbb{C},\omega\mapsto(-1)^{p+q+1}T(d\omega)\big).$$

Let us complete the definitions that first appeared in Section 2.2, which give several important examples.

- Let $\omega \in L^1 \otimes_{\mathscr{C}^\infty} \mathscr{A}^{p,q}$, then we can define a current $[\omega] \in \mathscr{D}^{p,q}$ induced by ω to be

$$[\omega](\eta) := \int_X \omega \wedge \eta, \quad \eta \in \mathscr{A}^{d-p,d-q}.$$

- Let Y be a codimension p irreducible **smooth** complex submanifold of X, then we get a **Dirac current** $\delta_Y \in \mathscr{D}^{p,p}$ defined by

$$\delta_Y(\eta) := \int_Y \eta, \quad \eta \in \mathscr{A}^{d-p,d-p}.$$

- More generally, for a codimension p irreducible complex submanifold Y (not necessarily smooth) of X with embedding $i : Y \hookrightarrow X$, define a current $\delta_Y \in \mathscr{D}^{p,p}$ by

$$\delta_Y(\eta) := \int_{\text{non-singular locus of } Y} i^*\eta, \quad \eta \in \mathscr{A}^{d-p,d-p}.$$

  The well-definedness of this δ_Y is given by the following Hironaka's theorem on the resolution of singularities.

  Theorem 3.2.2 (Hironaka)**.** *Given any $Z \subseteq Y$, where Z contains the singular locus of Y, there exists a proper map $\pi : \widetilde{Y} \to Y$ such that:*

  - *$\widetilde{Y}$ is smooth.*
  - *$\pi^{-1}(Z)$ is a divisor with normal crossings.*
  - *$\pi : \widetilde{Y} \setminus \pi^{-1}(Z) \to Y \setminus Z$ is an isomorphism.*
  - $\delta_Y(\eta) = \int_{Y \setminus Z} i^*\eta = \int_{\widetilde{Y}} \pi^* i^* \eta.$

  By linearity we extend this definition to arbitrary codimension p complex submanifolds.

One can check that:

Exercise 3.2.3. *There are some identities:*

- *$dd^c T(\omega) = -T(dd^c \omega)$, so $dd^c[\omega](\eta) = -[\omega](dd^c \eta)$.*
- *$d[\omega] = [d\omega]$ (use Stokes' formula). But $d^c[\omega] \neq [d^c \omega]$, so $dd^c[\omega] \neq [dd^c \omega]$.*

Recall Proposition 2.2.3 states that if g_P is a Green function (of logarithmic type) with respect to μ, then $dd^c[-g_P] + \delta_P = [\mu]$. Refer to this fact, we may define a class of special currents. First of all, let us review the definition of forms of logarithmic type, this is to get some moderate growing forms to make the integrals converge.

A smooth form ω on $X \setminus Y$ is said to be **of logarithmic type along** Y, if there exists a projective map $\pi : \widetilde{X} \to X$ such that $\pi^{-1}(Y)$ is a divisor with normal crossings, $\pi : \widetilde{X} \setminus \pi^{-1}(Y) \to X \setminus Y$ is smooth and ω is the direct image by π of a form α on $\widetilde{X} \setminus \pi^{-1}(Y)$ with the following property: near each $x \in \widetilde{X}$, let $z_1 \cdots z_k = 0$ be a local equation of $\pi^{-1}(Y)$, then there exists ∂ and $\overline{\partial}$ closed smooth forms α_i and a smooth form γ such that

$$\alpha = \sum_{i=1}^{k} \alpha_i \log |z_i|^2 + \gamma.$$

Definition 3.2.4 (Green Currents)**.** A **Green current** for a codimension p complex submanifold Y (not necessarily irreducible, but does not contain any irreducible components of X), is a current $g_Y \in \mathscr{D}^{p-1,p-1}$ such that $-dd^c g_Y + \delta_Y = [\omega]$ for some $\omega \in \mathscr{A}^{p,p}$.

Theorem 3.2.5. *If X is Kähler, then every $Y \subseteq X$ has a Green current. If g_1, g_2 are two Green currents for Y, then*

$$g_1 - g_2 = [\eta] + \partial S_1 + \overline{\partial} S_2$$

for some $\eta \in \mathscr{A}^{p-1,p-1}$, $S_1 \in \mathscr{D}^{p-2,p-1}$, $S_2 \in \mathscr{D}^{p-1,p-2}$. In particular, there exists a smooth form g_Y on $X \setminus Y$ of logarithmic type along Y such that $[g_Y]$ is a Green current for Y, i.e. $dd^c[-g_Y] + \delta_Y = [\omega]$ for some smooth form ω.

Proof. First, let us show that if $T \in \mathscr{D}^{p,p}$ with $T = dS$ for some current S, then $T = dd^c U$ with $U \in \mathscr{D}^{p-1,p-1}$. This is the so-called $\partial\overline{\partial}$-lemma.

Recall the **Hodge decomposition** states that for a Kähler manifold X we get adjoints $\partial^*, \overline{\partial}^*, d^*$ of $\partial, \overline{\partial}, d$ for the L^2-scalar product on forms (see Remark 2.1.9), and if $\mathbb{H}^{p,q} := \ker(\Delta_{\mathrm{dR}})$ denotes the space of **harmonic forms** then under the L^2-scalar product we have

$$\mathscr{A}^{p,q} = \mathbb{H}^{p,q} \oplus \mathrm{im}(d) \oplus \mathrm{im}(d^*) = \mathbb{H}^{p,q} \oplus \mathrm{im}(\partial) \oplus \mathrm{im}(\partial^*) = \mathbb{H}^{p,q} \oplus \mathrm{im}(\overline{\partial}) \oplus \mathrm{im}(\overline{\partial}^*),$$

(or there may be such a decomposition for $\mathscr{D}^{p,q}$). Hence, if write $T = \partial S + \overline{\partial} S$, by Hodge decomposition $S = h_1 + \partial x_1 + \partial^* y_1 = h_2 + \overline{\partial} x_2 + \overline{\partial}^* y_2$, so $\partial S = \partial\overline{\partial} x_2 + \partial\overline{\partial}^* y_2$ and $\overline{\partial} S = \overline{\partial}\partial x_1 + \overline{\partial}\partial^* y_1$. Thus

$$T = \overline{\partial}\partial x_1 + \partial\overline{\partial} x_2 + \partial\overline{\partial}^* y_2 + \overline{\partial}\partial^* y_1.$$

Now $dT = 0$ implies $\partial T = \overline{\partial} T = 0$, so $\partial\overline{\partial}\partial^* y_1 = 0$ and $\overline{\partial}\partial\overline{\partial}^* y_2 = 0$. Therefore

$$0 = \langle \partial\overline{\partial}\partial^* y_1, \overline{\partial} y_1 \rangle_{L^2} = -\langle \overline{\partial}\partial^* y_1, \overline{\partial}\partial^* y_1 \rangle_{L^2},$$

so $\overline{\partial}\partial^* y_1 = 0$, and similarly $\partial\overline{\partial}^* y_2 = 0$. Hence $T = \partial\overline{\partial}(x_2 - x_1)$, i.e. $U = 2\pi i(x_1 - x_2)$.

Here we only prove the existence of Green currents (we omit the proof of the existence of logarithmic type Green currents). By Stokes' formula we have $d\delta_Y = 0$, hence by Hodge decomposition we deduce $\delta_Y = [\omega] + dS$ for some $\omega \in \mathscr{A}^{p,p}$ and some current S. By $\partial\overline{\partial}$-lemma, we have $[\omega] - \delta_Y = -dS = -dd^c g$ for some $g \in \mathscr{D}^{p-1,p-1}$. □

Theorem 3.2.6 (Poincaré-Lelong Formula)**.** *Let $\mathscr{L}$ be a Hermitian line bundle on X with metric $\|\cdot\|$, suppose s is a meromorphic section of $\mathscr{L}$, then $-\log\|s\|^2 \in L^1$ hence induces $[-\log\|s\|^2] \in \mathscr{D}^{0,0}$. This is a Green current for* $\mathrm{div}(s)$*, in fact*

$$dd^c[\log\|s\|^2] + \delta_{\mathrm{div}(s)} = [c_1(\mathscr{L}, \|\cdot\|)].$$

Proof. Refer to the proof of Proposition 2.1.12. Note that $c_1(\mathscr{L}, \|\cdot\|) = -dd^c \log\|s\|^2$ for some meromorphic section s. After resolving the singularities, we may assume that in a local chart $U \xrightarrow{\sim} \mathbb{C}^d$, $\mathrm{div}(s)$ has equation $z_1 \cdots z_k = 0$. By linearity we are reduced to the case $s = z_1$. Since $-dd^c \log|z_1|^2 = 0$, apply Exercise 3.2.3, what we have to show is

$$\int_U \log|z_1|^2 dd^c \omega = \int_{z_1=0} \omega,$$

where $\omega \in \mathscr{A}^{d-1,d-1}$ with compact support in U. Indeed,

$$\begin{aligned}
&\int_U \log|z_1|^2 dd^c\omega \\
&= \lim_{r\to 0}\int_{|z_1|\geq r} \log|z_1|^2 \wedge dd^c\omega \\
&= \lim_{r\to 0}\int_{|z_1|=r,\curvearrowright} \log|z_1|^2 \wedge d^c\omega - \lim_{r\to 0}\int_{|z_1|\geq r} d\log|z_1|^2\wedge d^c\omega && \text{(Stokes' formula)} \\
&= \lim_{r\to 0}\int_{|z_1|\geq r} d^c\log|z_1|^2 \wedge d\omega && \text{(Exercise 2.2.4)} \\
&= \lim_{r\to 0}\int_{|z_1|\geq r} dd^c\log|z_1|^2 \wedge \omega - \lim_{r\to 0}\int_{|z_1|=r,\curvearrowright} d^c\log|z_1|^2\wedge \omega && \text{(Stokes' formula)} \\
&= \int_{z_1=0}\omega. && (dd^c\log|z_1|^2 = 0)
\end{aligned}$$

The last term is because $d^c\log|z_1|^2 = \frac{\partial-\overline{\partial}}{4\pi i}\log(z_1\overline{z_1}) = \frac{1}{2\pi}\mathrm{im}(\frac{dz_1}{z_1}) = \frac{d\arg(z_1)}{2\pi}$. □

As the end of this section, we introduce the $\star$-product of Green currents.

Definition 3.2.7 ($\star$-Product)**.** Let g_Y be a form of logarithmic type for Y given by Theorem 3.2.5 such that $dd^c[-g_Y]+\delta_Y = [\omega_Y]$, let g_Z be a Green current for Z. Define their **$\star$-product** to be

$$[g_Y]\star g_Z := [g_Y]\wedge\delta_Z + [\omega_Y]\wedge g_Z,$$

where $([g_Y]\wedge\delta_Z)(\eta) := \int_Z g_Y\wedge\eta$ (assume the singularities have been resolved) and $([\omega_Y]\wedge g_Z)(\eta) := g_Z(\omega_Y\wedge\eta)$.

This definition is well-defined, but not trivial. In contrast, the following fact is more important.

Proposition 3.2.8. *If Y,Z intersect properly, then*

$$-dd^c([g_Y]\star g_Z) = [\omega_Y\wedge\omega_Z] - \sum_x i_x(Y,Z)\delta_x,$$

where x runs out of all irreducible components of $Y\cap Z$.

Proof. We prove this formally. Indeed,

$$\begin{aligned}
dd^c([g_Y]\star g_Z) &= dd^c[g_Y]\wedge\delta_Z + [\omega_Y]\wedge dd^c g_Z \\
&= (\delta_Y - [\omega_Y])\wedge\delta_Z + [\omega_Y]\wedge(\delta_Z - [\omega_Z]) \\
&= \delta_Y\wedge\delta_Z - [\omega_Y]\wedge[\omega_Z] \\
&= -([\omega_Y\wedge\omega_Z] - \delta_{Y\cap Z}),
\end{aligned}$$

as desired. □

Remark 3.2.9. Let $Y\subseteq X$ be a closed irreducible submanifold and g_Y a Green current for Y. By Theorem 3.2.5, there exists a Green form $\widetilde{g_Y}$ of logarithmic type for Y such that

$$g_Y - [\widetilde{g_Y}] = [\eta] + \partial S_1 + \overline{\partial}S_2,$$

so every Green current for Y may be represented by a Green form of logarithmic type along Y modulo $\mathrm{im}(\partial)+\mathrm{im}(\overline{\partial})$, since η is a smooth form. Hence, if $Y,Z \subseteq X$ are closed irreducible submanifolds such that $Z \nsubseteq Y$ and g_Y (resp. g_Z) a Green current for Y (resp. Z), then we can define the $\star$-product of g_Y with g_Z by

$$g_Y \star g_Z := [\widetilde{g_Y}] \star g_Z \quad (\mathrm{mod}\ \mathrm{im}(\partial)+\mathrm{im}(\overline{\partial})).$$

One can show that this definition does not depend on the choice of $\widetilde{g_Y}$.

Under the assumption of Remark 3.2.9, the $\star$-product satisfy some operational laws.

Proposition 3.2.10. *After modulo* $\mathrm{im}(\partial)+\mathrm{im}(\overline{\partial})$*, the $\star$-product is commutative and associative.*

Proof. We also compute formally with currents as if they are forms. If $Y,Z,W \subseteq X$ are closed irreducible submanifolds meeting properly with currents g_Y, g_Z, g_W, respectively, then

$$\begin{aligned} g_Y \star g_Z &= g_Y \wedge \delta_Z + \omega_Y \wedge g_Z && (-dd^c g_Y + \delta_Y = [\omega_Y]) \\ &= g_Y \wedge \delta_Z + \delta_Y \wedge g_Z - dd^c g_Y \wedge g_Z \\ &= g_Y \wedge \delta_Z + \delta_Y \wedge g_Z - g_Y \wedge dd^c g_Z && \text{(general case of Exercise 2.2.5)} \\ &= g_Z \star g_Y, \end{aligned}$$

and

$$g_Y \star (g_Z \star g_W) = g_Y \wedge \delta_Z \wedge \delta_W + \omega_Y \wedge g_Z \wedge \delta_W + \omega_Y \wedge \omega_Z \wedge g_W = (g_Y \star g_Z) \star g_W.$$

The strict proof uses the precise form of Hironaka's theorem on the resolution of singularities. □

3.3 Gillet-Soulé Intersection Pairing

In this section, we develop Arakelov geometry in higher dimensions. Let X be an integral regular projective flat scheme over $\mathbb{Z}$ with smooth generic fiber (i.e. an arithmetic variety), we will define the higher arithmetic Chow groups, and study the arithmetic intersection theory. The methods in which these theories are established are quite different from Chapter 2.3, but we will assert that these seemingly different geometries are essentially the same.

Let us introduce some notations. Assume X is an arithmetic variety over $\mathbb{Z}$, denote the complex conjugation by $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$, it is a continuous involution of $X(\mathbb{C})$. Put

- $A^{p,p}(X) := \{\omega \in \mathscr{A}^{p,p}(X(\mathbb{C})) : \omega \text{ real}, F_\infty^* \omega = (-1)^p \omega\}$.
- $\widetilde{A}^{p,p}(X) := A^{p,p}(X)/(\mathrm{im}(\partial)+\mathrm{im}(\overline{\partial}))$.
- $Z^{p,p}(X) := \ker\left(d : A^{p,p}(X) \to \mathscr{A}^{2p+1}(X(\mathbb{C}))\right) \subseteq A^{p,p}(X)$.
- $H^{p,p}(X) := \ker\left(dd^c : A^{p,p}(X) \to A^{p+1,p+1}(X)\right)/(\mathrm{im}(\partial)+\mathrm{im}(\overline{\partial})) \subseteq \widetilde{A}^{p,p}(X)$.
- $D^{p,p}(X) := \{T \in \mathscr{D}^{p,p}(X(\mathbb{C})) : T \text{ real}, F_\infty^* T = (-1)^p T\}$.

The notations above fit into the following diagram:

$$\begin{array}{ccccc} Z^{p,p}(X) & \hookrightarrow & A^{p,p}(X) & \xrightarrow{\omega\mapsto[\omega]} & D^{p,p}(X) \\ {\scriptstyle \omega\mapsto\omega(\text{mod im}(\partial)+\text{im}(\overline{\partial}))}\big\downarrow{\scriptstyle\gamma} & & \big\downarrow & & \\ H^{p,p}(X) & \hookrightarrow & \widetilde{A}^{p,p}(X) & & \end{array}$$

Now we give the definition of higher arithmetic Chow groups.

Definition 3.3.1 (Arithmetic Chow Groups)**.** Let X be an arithmetic variety over $\mathbb{Z}$. Define the group of **p-arithmetic cycles** to be

$$\widehat{Z}^p(X) := \left\{(Z, g_Z) : Z \in Z^p(X), g_Z \in \mathscr{D}^{p-1,p-1}(X(\mathbb{C})) \text{ a Green current for } Z(\mathbb{C})\right\},$$

with addition defined componentwise. Let $\widehat{R}^p(X) \subseteq \widehat{Z}^p(X)$ be the subgroup generated by pairs

$$\left(0, \partial(u) + \overline{\partial}(v)\right) \text{ and } \left(\text{div}(f), -[\log|f_{\mathbb{C}}|^2]\right),$$

where u (resp. v) is a current of type $(p-2, p-1)$ (resp. $(p-1, p-2)$), $f \in k(y)^\times$ for some $y \in X^{(p-1)}$, and $f_{\mathbb{C}}$ is the pull-back of f by $X(\mathbb{C}) \to X$. The quotient group $\widehat{\text{CH}}^p(X) := \widehat{Z}^p(X)/\widehat{R}^p(X)$ is called the p-th **arithmetic Chow group**.

Since X is projective, one can choose a "canonical volume form" such that $X(\mathbb{C})$ has finite volume. Hence, in the definition above, after resolving the singularities if needed, $-\log|f_{\mathbb{C}}|^2$ is a Lebesgue integrable function on $y(\mathbb{C})$ (because there are as many zeros and poles) and induces a Green current

$$-[\log|f_{\mathbb{C}}|^2] \in \mathscr{D}^{p-1,p-1}(X(\mathbb{C})), \quad \omega \mapsto \int_{y(\mathbb{C})} -\log|f_{\mathbb{C}}|^2 \wedge (\omega|_{y(\mathbb{C})})$$

for $\text{div}(f)(\mathbb{C})$ by Theorem 3.2.6.

Theorem 3.3.2. *There are two exact sequences:*

- $H^{p-1,p-1}(X) \xrightarrow{\alpha} \widehat{\text{CH}}^p(X) \xrightarrow{(\phi,\psi)} \text{CH}^p(X) \oplus Z^{p,p}(X).$
- $\widetilde{A}^{p-1,p-1}(X) \xrightarrow{\alpha} \widehat{\text{CH}}^p(X) \xrightarrow{\phi} \text{CH}^p(X) \longrightarrow 0.$

If we assume $-dd^c g_Z + \delta_Z = [\omega_Z]$ for $Z \in Z^p(X)$, then the maps are $\alpha : \omega \mapsto [(0, [\omega])]$; $\phi : [(Z, g_Z)] \mapsto [Z]$; $\psi : [(Z, g_Z)] \mapsto \omega_Z$.

Proof. We only prove the first one, all maps are well-defined. Indeed, α is well-defined because $\omega \in H^{p-1,p-1}(X)$ always a Green current for the zero cycle (this also implies $(\phi, \psi) \circ \alpha = 0$); ψ is well-defined because the Stokes' formula implies $d\delta_Z = 0$, so ω_Z is closed. To show $\ker(\phi, \psi) \subseteq \text{im}(\alpha)$, we know $(\phi, \psi)[(Z, g_Z)] = 0$ if and only if $-dd^c g_Z + \delta_Z = 0$ for some $Z = \sum_y \text{div}(f_y)$, where $f_y \in k(y)^\times$, $y \in X^{(p-1)}$. Thus

$$[(Z, g_Z)] = \left[\left(\sum_y \text{div}(f_y), g_Z\right)\right] = \left[\left(0, g_Z + \sum_y [\log|f_{y,\mathbb{C}}|^2]\right)\right] =: [(0, G)].$$

Since $-dd^c G = -dd^c g_Z + \delta_Z = 0$, so G and 0 are Green currents for the zero cycle. By Theorem 3.2.5 we have $G = [\eta] + \partial S_1 + \overline{\partial} S_2$ for some smooth form η, one can verify $\alpha(\eta) = [(Z, g_Z)]$. □

Remark 3.3.3. In fact, some extra terms can be added to the left and the right of the exact sequences in Theorem 3.3.2, the following sequences are exact:

$$E_{2,X}^{p-1,-p}(X) \to H^{p-1,p-1}(X) \to \widehat{\mathrm{CH}}^p(X) \to \mathrm{CH}^p(X) \oplus Z^{p,p}(X) \to H^{p,p}(X) \to 0,$$

and

$$E_{2,X}^{p-1,-p}(X) \to \widetilde{A}^{p-1,p-1}(X) \to \widehat{\mathrm{CH}}^p(X) \to \mathrm{CH}^p(X) \to 0.$$

The definition of $E_{2,X}^{p-1,-p}(X)$ can be found in the proof of Theorem 3.1.8, each element $\big(f_y : \sum_y \mathrm{div}(f_y) = 0\big) \in E_{2,X}^{p-1,-p}(X)$ is sent to $-\sum_y \log|f_{y,\mathbb{C}}|^2 \in H^{p-1,p-1}(X)$. The map

$$\mathrm{CH}^p(X) \oplus Z^{p,p}(X) \to H^{p,p}(X), \quad ([Z], \omega) \mapsto \mathrm{cyc}([Z]) - \omega \quad (\mathrm{mod}\ \mathrm{im}(\partial) + \mathrm{im}(\overline{\partial}))$$

is defined as follows.

For each $0 \le p \le \dim(X)$, we can construct a homomorphism

$$\mathrm{cyc} : \mathrm{CH}^p(X) \to \mathrm{CH}^p(X(\mathbb{C})) \to H^{2p}(X(\mathbb{C}), \mathbb{Z}) \to H^{p,p}(X).$$

More precisely, let $Z \subseteq X$ be a (generically smooth, without loss of generality) closed subvariety of codimension p and $U := X \setminus Z$, note that $X(\mathbb{C})$ is a smooth projective variety over $\mathbb{C}$, we have an exact sequence by excision theorem:

$$\cdots \to H^{2p}((X(\mathbb{C}); U(\mathbb{C})), \mathbb{Z}) \xrightarrow{\varepsilon} H^{2p}(X(\mathbb{C}), \mathbb{Z}) \to H^{2p}(U(\mathbb{C}), \mathbb{Z}) \to \cdots.$$

Recall the Thom isomorphism states

$$\mathrm{Th} : H^{2p}((X(\mathbb{C}); U(\mathbb{C})), \mathbb{Z}) \xrightarrow{\sim} H^0(Z(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}^{\text{the number of irreducible components of } Z(\mathbb{C})},$$

use these we define $\mathrm{cyc}([Z]) := \varepsilon \circ \mathrm{Th}^{-1}(1, \cdots, 1)$.

Example 3.3.4. In the case of curves, let $X = \mathrm{Spec}(\mathscr{O}_K)$ for a number field K, then Theorem 3.3.2 (or Remark 3.3.3) becomes

$$\Big(0 \to \mu_K \to \Big)\, \mathscr{O}_K^\times \to \mathbb{R}^{r_1 + \frac{r_2}{2}} \to \widehat{\mathrm{Pic}}(X) \to \mathrm{Cl}(\mathscr{O}_K) \to 0 \to 0,$$

where r_1 (resp. r_2) is the number of real (resp. complex) embeddings. All details have been defined in Chapter 1.1.

We now introduce Gillet-Soulé's arithmetic intersection theory.

Theorem 3.3.5 (Gillet-Soulé)**.** *Let X be an arithmetic variety over $\mathbb{Z}$, then there exists a bilinear pairing*

$$(\cdot,\cdot) : \widehat{\mathrm{CH}}^p(X) \times \widehat{\mathrm{CH}}^q(X) \to \widehat{\mathrm{CH}}^{p+q}(X)_{\mathbb{Q}},$$

It turns $\bigoplus_p \widehat{\mathrm{CH}}^p(X)_{\mathbb{Q}}$ into a commutative graded $\mathbb{Q}$-algebra with unit $[(X,0)] \in \widehat{\mathrm{CH}}^0(X)$. Moreover,

- $\phi\big([(Z,g_Z)], [(W,g_W)]\big) = \big(\phi[(Z,g_Z)], \phi[(W,g_W)]\big) = ([Z],[W])$.
- $\psi\big([(Z,g_Z)], [(W,g_W)]\big) = \psi[(Z,g_Z)] \wedge \psi[(W,g_W)]$.

Proof. (proof sketch). Suppose we have two arithmetic cycles (Z, g_Z) and (W, g_W), our aim is to define $[(Z \cap W, g_{Z\cap W})]$. Indeed, the cycle $Z \cap W$ is given by Theorem 3.1.4 algebraically. Now let

$$\mathrm{CH}^p_{\mathrm{fin}}(X) := \frac{\{Z \in Z^p(X) : \mathrm{Supp}(Z) \cap X_{\mathbb{Q}} = \emptyset\}}{\langle \mathrm{div}(f) : f \in k(y)^\times \text{ for some } y \in X^{(p-1)} \setminus X_{\mathbb{Q}} \rangle},$$

there is a canonical map

$$Z^p(X) \to \mathrm{CH}^p(X) \to \mathrm{CH}^p_{\mathrm{fin}}(X) \oplus Z^p(X_{\mathbb{Q}}), \quad Z \mapsto Z_{\mathrm{fin}} + Z_{\mathbb{Q}}.$$

To define $g_{Z\cap W}$, one just needs the generic part $Z^{\cdot}(X_{\mathbb{Q}})$, because the finite part $\mathrm{CH}^{\cdot}_{\mathrm{fin}}(X)$ does not produce cycles in $X(\mathbb{C})$. We can assume that $Z_{\mathbb{Q}}$ and $W_{\mathbb{Q}}$ intersect properly (use the $\mathbb{Q}$-moving lemma), then Theorem 3.1.4 implies there exists a well-defined intersection cycle $(Z_{\mathbb{Q}}, W_{\mathbb{Q}})$ in $X_{\mathbb{Q}}$. The current for $Z \cap W$ is now defined to be $g_{Z_{\mathbb{Q}}(\mathbb{C})} \star g_{W_{\mathbb{Q}}(\mathbb{C})}$, since Proposition 3.2.8 guarantees it is a Green current for $(Z \cap W)(\mathbb{C})$. □

In the proof of Theorem 3.3.5, we decompose the so-called intersection pairing into the finite part and the generic part, in order to correspond with the vertical divisors and horizontal divisors in Arakelov's intersection theory in Chapter 2.3. These two types of intersections should be treated differently. We use an exercise to summarize this.

Exercise 3.3.6. *Let X be an arithmetic surface over $\mathbb{Z}$ and assume μ is a canonical volume form on $X(\mathbb{C})$. Show that there is an embedding*

$$\widehat{\mathrm{CH}}^1(X, \mu) \hookrightarrow \widehat{\mathrm{CH}}^1(X), \quad (Z, r) \mapsto \big(Z, [g_{Z(\mathbb{C})} + 2r]\big),$$

where $g_{Z(\mathbb{C})}$ is the Green function of logarithmic type with respect to μ. Try to reconstruct the Arakelov intersection theory (Theorem 2.3.4) by using the Gillet-Soulé intersection theory (Theorem 3.3.5). For the finite part, you should define a natural degree map from the algebraic Chow group to $\mathbb{R}$; for the infinite part, you should use $\star$-product to define intersect currents and compute their values at the constant function $1/2$.

Theorem 3.3.7. *Let X, Y be arithmetic varieties, and let $f : Y \to X$ be a morphism.*

- *There is a pull-back homomorphism $f^* : \widehat{\mathrm{CH}}^p(X) \to \widehat{\mathrm{CH}}^p(Y)_{\mathbb{Q}}$, it is multiplicative.*
- *If f is proper, $f_{\mathbb{Q}} : Y_{\mathbb{Q}} \to X_{\mathbb{Q}}$ is smooth and X, Y are equidimensional, then there is a push-forward homomorphism $f_* : \widehat{\mathrm{CH}}^p(Y) \to \widehat{\mathrm{CH}}^{p-(\dim(Y)-\dim(X))}(X)$.*
- *The projection formula holds:*

$$f_*(f^*\alpha, \beta) = (\alpha, f_*\beta), \quad \textit{for } \alpha \in \widehat{\mathrm{CH}}^p(X), \beta \in \widehat{\mathrm{CH}}^q(Y).$$

- *$(\cdot)^*$ is a contravariant functor, $(\cdot)_*$ is a covariant functor.*

Proof. (proof sketch). We only need to prove the first three items.

- Let $[(Z,g_Z)] \in \widehat{\mathrm{CH}}^p(X)$ where Z is irreducible and assume $\mathrm{codim}_{Y_\mathbb{Q}}(f^{-1}(Z)_\mathbb{Q}) = p$ (any general case can be reduced to this case). By functoriality of K-theory or Chow theory (Theorem 3.1.4), we form a class $f^*[Z] \in \mathrm{CH}^p(Y)_\mathbb{Q}$, its image under

$$\mathrm{CH}^p(Y)_\mathbb{Q} \to \mathrm{CH}^p_{\mathrm{fin}}(Y)_\mathbb{Q} \oplus Z^p(Y_\mathbb{Q})_\mathbb{Q}$$

is also denoted by $f^*[Z]$. Furthermore, one can verify the pull-back $f^*_\mathbb{C}(g_Z)$ is also a Green current. So

$$f^*[(Z,g_Z)] := [(f^*[Z], f^*_\mathbb{C}(g_Z))] \in \widehat{\mathrm{CH}}^p(Y)_\mathbb{Q}.$$

- Denote $q := \dim(Y) - \dim(X)$, we construct the map $\widehat{Z}^p(Y) \to \widehat{Z}^{p-q}(X)$, given by $(Z,g_Z) \mapsto (f_*(Z), f_{\mathbb{C},*}(g_Z))$, where

$$f_*(Z) := \begin{cases} [k(Z):k(f(Z))]\cdot f(Z) & \dim(f(Z)) = \dim(Z) \\ 0 & \dim(f(Z)) < \dim(Z) \end{cases}$$

To study $f_{\mathbb{C},*}(g_Z)$, observe that for a differential form η on $X(\mathbb{C})$ of appropriate degree, we have

$$\begin{aligned} & (f_{\mathbb{C},*}(\delta_Z))(\eta) \\ =& \int_{Z(\mathbb{C})} f^*_\mathbb{C}(\eta|_{f(Z(\mathbb{C}))}) \\ =& \begin{cases} [k(Z(\mathbb{C})):k(f(Z(\mathbb{C})))]\cdot \int_{f(Z(\mathbb{C}))}\eta & \dim(f(Z(\mathbb{C}))) = \dim(Z(\mathbb{C})) \\ 0 & \dim(f(Z(\mathbb{C}))) < \dim(Z(\mathbb{C})) \end{cases} \end{aligned}$$

Hence $f_{\mathbb{C},*}(\delta_Z) = \delta_{f_*(Z)}$, from which we deduce

$$-dd^c(f_{\mathbb{C},*}(g_Z)) = [f_{\mathbb{C},*}(\omega_Z)] - \delta_{f_*(Z)}.$$

So the map we have constructed is reasonable. It is easy to see this map sends $\widehat{R}^p(Y)$ into $\widehat{R}^{p-q}(X)$, because for a rational function $h \in k(W)^\times$ with $\dim(f(W)) = \dim(W)$ for some $W \in Y^{(p-1)}$, one can check the image of $\big(\mathrm{div}(h), -[\log|h_\mathbb{C}|^2]\big)$ is

$$\Big(\mathrm{div}\big(\mathrm{Nm}_{k(W)/k(f(W))}(h)\big), -\big[\log\big|\mathrm{Nm}_{k(W(\mathbb{C}))/k(f(W(\mathbb{C})))}(h_\mathbb{C})\big|^2\big]\Big).$$

For the third item, we already have the projection formula for algebraic cycles, see Proposition 3.1.6. Therefore, we are left to prove it for Green currents. This can be calculated by the definition of the $\star$-product, we leave it as an exercise. □

3.4 Characteristic Classes

In this section, we establish the theory of Chern-Weil, and use them to study metrized vector bundles at infinite parts of arithmetic varieties.

Let us review Chern-Weil theory first. Let E be a rank r Hermitian vector bundle on a complex manifold X (we discard the symbols $\|\cdot\|$ or h for simplicity), suppose its curvature matrix is Ω, then the **total Chern class** of E is

$$\det\left(I+\frac{i}{2\pi}\Omega\right)=1+c_1(E)+\cdots+c_r(E)\in H^{\cdot}_{\mathrm{dR}}(X),$$

where $c_i(E)\in H^{2i}_{\mathrm{dR}}(X)$ is the i-th Chern class.

Definition 3.4.1 (Chern Character Forms)**.** Let E be a rank r Hermitian vector bundle on X. Consider the **Chern polynomial**

$$\mathrm{ch}(x_1,\cdots,x_r):=\sum_{j=1}^{r}e^{t_j}=r+\mathrm{ch}_1(x_1,\cdots,x_r)+\mathrm{ch}_2(x_1,\cdots,x_r)+\cdots,$$

where

$$\mathrm{ch}_k(x_1,\cdots,x_r):=\sum_{i=1}^{r}\frac{t_i^k}{k!}$$

and $x_1,\cdots,x_r$ are elementary symmetric polynomials in $t_1,\cdots,t_r$. Then the **Chern character** of E is defined to be a differential form

$$\mathrm{ch}(E):=\mathrm{ch}(c_1(E),\cdots,c_r(E))\in\bigoplus_{p\geq 0}\mathscr{A}^{p,p}.$$

Exercise 3.4.2. *If the total Chern class has form* $\prod_i(1+t_i)$*, then* $\mathrm{ch}(E)=\sum_i e^{t_i}$*. In particular, if* $\mathscr{L}$ *is a Hermitian line bundle, then* $\mathrm{ch}(\mathscr{L})=e^{c_1(\mathscr{L})}$.

There are some basic properties of Chern characters, but we do not intend to prove them here.

Proposition 3.4.3. *Let* $f:Y\to X$ *be a holomorphic map, let* E,F *be Hermitian vector bundles on* X*. Then*

- ch *depends on the choice of the metric on E, but not its cohomology class.*
- ch *is a characteristic class, i.e.* $f^*\mathrm{ch}(E)=\mathrm{ch}(f^*E)$.
- ch *satisfies Whitney's formula, i.e.* $\mathrm{ch}(E\oplus F)=\mathrm{ch}(E)+\mathrm{ch}(F)$ *and* $\mathrm{ch}(E\otimes F)=\mathrm{ch}(E)\wedge\mathrm{ch}(F)$.

Whitney's formula reminds us that Chern characters are only additive for split exact sequences, so we can use them to construct an invariant to measure the splitness of exact sequences of Hermitian vector bundles.

Theorem 3.4.4. *There is a unique way to attach to every sequence of Hermitian vector bundles*

$$\mathfrak{E}:\quad 0\to E'\xrightarrow{f}E\xrightarrow{g}E''\to 0$$

a form

$$\widetilde{\mathrm{ch}}(\mathfrak{E})\in\bigoplus_{p\geq 0}\left(\mathscr{A}^{p,p}/(\mathrm{im}(\partial)+\mathrm{im}(\overline{\partial}))\right),$$

called the ***Bott-Chern (secondary) character form*** *of* $\mathfrak{E}$*, satisfying the following properties:*

- $dd^c\,\widetilde{\mathrm{ch}}(\mathfrak{E}) = \mathrm{ch}(E') - \mathrm{ch}(E) + \mathrm{ch}(E'')$.

- *For every holomorphic map $f : Y \to X$ of complex manifolds, we have $f^*\,\widetilde{\mathrm{ch}}(\mathfrak{E}) \equiv \widetilde{\mathrm{ch}}(f^*\mathfrak{E})$.*

- *If $\mathfrak{E}$ is split, then $\widetilde{\mathrm{ch}}(\mathfrak{E}) \equiv 0$.*

- *If F is a Hermitian vector bundle, then $\widetilde{\mathrm{ch}}(\mathfrak{E} \otimes F) \equiv \widetilde{\mathrm{ch}}(\mathfrak{E}) \wedge \mathrm{ch}(F)$.*

- *If $\mathfrak{E}_1$, $\mathfrak{E}_2$ are two exact sequences of Hermitian vector bundles, then $\widetilde{\mathrm{ch}}(\mathfrak{E}_1 \oplus \mathfrak{E}_2) \equiv \widetilde{\mathrm{ch}}(\mathfrak{E}_1) + \widetilde{\mathrm{ch}}(\mathfrak{E}_2)$.*

Proof. (proof sketch).

- First, we construct a suitable Hermitian vector bundle $(\widetilde{E}, \widetilde{h})$ on $X \times \mathbb{P}^1(\mathbb{C})$ which describes the process of deformation of E to $E' \oplus E''$ on X.

  Let $\mathscr{O}(1)$ be the degree one line bundle on $\mathbb{P}^1(\mathbb{C})$ with the metric given by Exercise 2.1.13, let $\gamma \in \Gamma(\mathbb{P}^1(\mathbb{C}), \mathscr{O}(1))$ be a holomorphic section which has a single zero at ∞. Consider the Hermitian vector bundle $E'(1) := E' \otimes \mathscr{O}(1)$ on $X \times \mathbb{P}^1(\mathbb{C})$ with metric given by E' and $\mathscr{O}(1)$, define $\widetilde{E} := (E \oplus E'(1))/E'$, where

  $$E' \hookrightarrow \widetilde{E}, \quad s \mapsto (-f(s), s \otimes (\gamma \cdot s)).$$

  Now we have the following commutative diagram of vector bundles on $X \times \mathbb{P}^1(\mathbb{C})$:

  $$\begin{array}{ccccccccc}
\mathfrak{E}: & 0 & \longrightarrow & E' & \xrightarrow{\;f\;} & E & \xrightarrow{\;g\;} & E'' & \longrightarrow & 0 \\
 & & & \Big\downarrow {\scriptstyle s \overset{\mathrm{id}\otimes\gamma}{\longmapsto} s\otimes(\gamma\cdot s)} & & \Big\downarrow {\scriptstyle t \overset{\rho}{\mapsto} [(t,0)]} & & \Big\| {\scriptstyle \cong} & & \\
\mathfrak{F}: & 0 & \longrightarrow & E'(1) & \xrightarrow[s\otimes r \overset{\theta}{\mapsto} [(0,s\otimes r)]]{} & \widetilde{E} & \xrightarrow[{[(t,s\otimes r)]} \overset{\vartheta}{\mapsto} g(t)]{} & E'' & \longrightarrow & 0
\end{array}$$

  For every point $z \in \mathbb{P}^1(\mathbb{C})$, denote $i_z : X \to X \times \mathbb{P}^1(\mathbb{C})$, $x \mapsto (x,z)$. Then check by stalks we know:

  - $i_0^*\widetilde{E} \cong E$. This is because now $\mathrm{id} \otimes \gamma$ is an isomorphism, so the conclusion is followed by the five lemma.
  - $i_\infty^*\widetilde{E} \cong E' \oplus E''$. This is because now $\mathrm{id} \otimes \gamma : s \mapsto 0$, so $i_\infty^*\widetilde{E} \cong \big(i_\infty^*(E/E') \oplus i_\infty^* E'(1)\big)$, but $i_\infty^* E'(1) \cong E'$.

  To make $\widetilde{E}$ a Hermitian vector bundle, use a partition of unity, we can set a Hermitian metric $\widetilde{h}$ on $\widetilde{E}$ such that the isomorphisms $i_0^*\widetilde{E} \cong E$ and $i_\infty^*\widetilde{E} \cong E' \oplus E''$ become isometries.

- Assume the Bott-Chern character forms are exist in any case. Consider the integral

  $$I := \int_{\mathbb{P}^1(\mathbb{C})} \log|z|^2 \wedge dd^c\,\widetilde{\mathrm{ch}}(\mathfrak{F}).$$

One can compute

$$\begin{aligned}I &= \int_{\mathbb{P}^1(\mathbb{C})} \log|z|^2 \wedge \Big(\mathrm{ch}(E'(1)) - \mathrm{ch}(\widetilde{E}) + \mathrm{ch}(E'')\Big)\\ &= \mathrm{ch}(E')\int_{\mathbb{P}^1(\mathbb{C})} \log|z|^2 \wedge \mathrm{ch}(\mathscr{O}(1)) - \int_{\mathbb{P}^1(\mathbb{C})} \log|z|^2 \wedge \mathrm{ch}(\widetilde{E}) + \mathrm{ch}(E'')\int_{\mathbb{P}^1(\mathbb{C})} \log|z|^2\\ &= -\int_{\mathbb{P}^1(\mathbb{C})} \log|z|^2 \wedge \mathrm{ch}(\widetilde{E}).\end{aligned}$$

The first integral vanishes since $\mathrm{ch}(\mathscr{O}(1)) = 1 + \frac{i}{2\pi}\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2}$, and the integral of the top degree of $\log|z|^2 \wedge \mathrm{ch}(\mathscr{O}(1))$ is 0. On the other hand, use the same argument in the proof of Theorem 3.2.6,

$$\begin{aligned}I &= \int_{z=0} \widetilde{\mathrm{ch}}(\mathfrak{F}) - \int_{z=\infty} \widetilde{\mathrm{ch}}(\mathfrak{F}) && \text{(Theorem 3.2.6's proof)}\\ &= i_0^*\widetilde{\mathrm{ch}}(\mathfrak{F}) - i_\infty^*\widetilde{\mathrm{ch}}(\mathfrak{F}) && \Big(\int_{z=z_0}\omega(x,z) = \omega(x,z)\big|_{(X,z_0)}\Big)\\ &\equiv \widetilde{\mathrm{ch}}(i_0^*\mathfrak{F}) - \widetilde{\mathrm{ch}}(i_\infty^*\mathfrak{F}) \quad (\mathrm{mod}\ \mathrm{im}(\partial)+\mathrm{im}(\overline{\partial})) && (i_z^*\circ\widetilde{\mathrm{ch}} \equiv \widetilde{\mathrm{ch}}\circ i_z^*)\\ &\equiv \widetilde{\mathrm{ch}}(\mathfrak{E}) \quad (\mathrm{mod}\ \mathrm{im}(\partial)+\mathrm{im}(\overline{\partial})). && (i_0^*\mathfrak{F} = \mathfrak{E};\ i_\infty^*\mathfrak{F}\ \text{splits})\end{aligned}$$

- To show the existence, note that $\widetilde{\mathrm{ch}}(\mathfrak{E})$ can be defined to be a integral

  $$\widetilde{\mathrm{ch}}(\mathfrak{E}) := -\int_{\mathbb{P}^1(\mathbb{C})} \log|z|^2 \wedge \mathrm{ch}(\widetilde{E},\widetilde{h}),$$

  so we are done. One can check this definition does not depend on the construction of $\widetilde{h}$, and hence it implies the uniqueness of $\widetilde{\mathrm{ch}}$ modulo $\mathrm{im}(\partial)+\mathrm{im}(\overline{\partial})$.

All that remains is to verify that this construction satisfies the conditions above. □

Like the previous processes, Chern classes and Chern characters also have arithmetic analogies in Arakelov geometry. More generally, for any characteristic class, it always has arithmetic analogues.

For example, here we construct the arithmetic Chern character $\widehat{\mathrm{ch}}$. We hope this invariant satisfies properties similar to the properties in Proposition 3.4.3 and Theorem 3.4.4, so during the construction process we can assume it already satisfies these properties.

Refer to Exercise 3.4.2, we shall start with the case of metrized line bundles first.

Definition 3.4.5 (Arithmetic Chern Characters For Line Bundles)**.** Let X be an arithmetic variety over $\mathbb{Z}$ and $\mathscr{L}$ a metrized line bundle on X. Define the **arithmetic Chern character** for $\mathscr{L}$ to be

$$\widehat{\mathrm{ch}}(\mathscr{L}) := \exp\big(\big[(\mathrm{div}(s), -[\log\|s_{\mathbb{C}}\|^2])\big]\big) \in \widehat{\mathrm{CH}}^{\cdot}(X)_{\mathbb{Q}},$$

where s is a non-zero rational section of $\mathscr{L}$.

Exercise 3.4.6. *The map ψ in Theorem 3.3.2 maps $\widehat{\mathrm{ch}}(\mathscr{L})$ to $\mathrm{ch}(\mathscr{L}_{\mathbb{C}})$.*

Next, for a general Hermitian vector bundle, we will use a more advanced version of the **splitting principle** to descend the dimension. That is, we should introduce the construction of arithmetic Chern characters for some special bundles (indeed, tautological bundles) on Grassmannians, and then use the fact that any bundle on a projective variety, once tensored with an ample line bundle, is classified by a map to a Grassmannian.

Let m,n be positive integers. Recall the (m,n)**-Grassmannian** over $\mathbb{Z}$ is a scheme $\mathrm{Gr}_{m,n}$ associated by a representable contravariant functor

$$\mathrm{Hom}_{\mathbb{Z}\text{-Sch}}(\cdot,\mathrm{Gr}_{m,n}):\mathbb{Z}\text{-Sch}\longrightarrow\text{Sets},\quad X\longmapsto\left\{\mathscr{O}_X^{m+n}\overset{f}{\twoheadrightarrow}\mathscr{F}:\ \mathscr{F}\text{ is a rank }n\text{ bundle on }X\right\}.$$

In particular, if we take $X=\mathrm{Gr}_{m,n}$ itself, then the vector bundle on $\mathrm{Gr}_{m,n}$ corresponding to $\mathrm{id}_{\mathrm{Gr}_{m,n}}$ is the special bundle we want, denoted as $\mathrm{Taut}_{m,n}$.

There is a natural map, which comes from the natural transformation

$$\mu_{q,n}:(\mathrm{Gr}_{q,1})^n\longrightarrow\mathrm{Gr}_{qn,n},\quad\left(\mathscr{O}_X^{q+1}\twoheadrightarrow\mathscr{L}_i\right)_{1\le i\le n}\longmapsto\left(\mathscr{O}_X^{qn+n}\twoheadrightarrow\bigoplus_{i=1}^n\mathscr{L}_i\right).$$

The splitting principle states that $\mu_{q,n}^*(\mathrm{Taut}_{qn,n})=\bigoplus_{i=1}^n\mathscr{L}_i$ for some line bundles $\mathscr{L}_i$ on $(\mathrm{Gr}_{q,1})^n$. This is because $\mu_{q,n}^*(\mathrm{Taut}_{qn,n})$ is corresponding to

$$\mathrm{id}_{\mathrm{Gr}_{qn,n}}\circ\mu_{q,n}\in\mathrm{Hom}_{\mathbb{Z}\text{-Sch}}\big((\mathrm{Gr}_{q,1})^n,\mathrm{Gr}_{qn,n}\big)\cong\prod_{i=1}^n\mathrm{Hom}_{\mathbb{Z}\text{-Sch}}(\mathrm{Gr}_{q,1},\mathrm{Gr}_{qn,n}).$$

To do Arakelov geometry, we should equip $\mathrm{Taut}_{qn,n,\mathbb{C}}$ with a natural Hermitian metric. Like Exercise 2.1.13, we equip it with a quotient metric induced by the metric $\sum_{i=1}^{qn+n}|z_i|^2$ on $\mathbb{C}^{qn+n}$. One can check this metric is compatible with the decomposition

$$\mu_{q,n,\mathbb{C}}^*(\mathrm{Taut}_{qn,n,\mathbb{C}})=\bigoplus_{i=1}^n\mathscr{L}_{i,\mathbb{C}},$$

if all $\mathscr{L}_{i,\mathbb{C}}$ have suitable metrics.

In order for $\widehat{\mathrm{ch}}$ to satisfy properties in Proposition 3.4.3, according to Definition 3.4.5, we make the following definition.

Definition 3.4.7 (Arithmetic Chern Characters For Tautological Bundles)**.** Suppose q is big enough (this is a technical requirement). Define the arithmetic Chern character for the metrized vector bundle $\mathrm{Taut}_{qn,n}$ to be

$$\widehat{\mathrm{ch}}(\mathrm{Taut}_{qn,n}):=(\mu_{q,n}^*)^{-1}\left(\sum_{i=1}^n\widehat{\mathrm{ch}}(\mathscr{L}_i)\right)\in\widehat{\mathrm{CH}}^{\cdot}(\mathrm{Gr}_{qn,n})_{\mathbb{Q}}.$$

Finally, for the general case, let E be a rank n metrized vector bundle on an arithmetic variety X defined over $\mathbb{Z}$. Since X is projective, there exists an ample line bundle on X. This means there exists a line bundle $\mathscr{L}$ on X such that $E\otimes\mathscr{L}^{-1}$ is generated by global sections, i.e. there is a surjective map $\mathscr{O}_X^N\twoheadrightarrow E\otimes\mathscr{L}^{-1}$ for some N. Indeed we can take $N=qn+n$. By the definition of Grassmannian, there exists a morphism $f:X\to\mathrm{Gr}_{qn,n}$ corresponding to $\mathscr{O}_X^N\twoheadrightarrow E\otimes\mathscr{L}^{-1}$, i.e. $f^*(\mathrm{Taut}_{qn,n})\cong E\otimes\mathscr{L}^{-1}$. We can equip $\mathscr{L}$ with a suitable metric to make this isomorphism an isometry at the infinite part.

Proposition 3.4.8. *Suppose we already have a complete definition for* $\widehat{\mathrm{ch}}$*. Let*

$$\mathfrak{E}: \quad 0 \to E' \to E \to E'' \to 0$$

be an exact sequence of metrized vector bundles on X, then

$$\widehat{\mathrm{ch}}(E') - \widehat{\mathrm{ch}}(E) + \widehat{\mathrm{ch}}(E'') = \alpha(\widetilde{\mathrm{ch}}(\mathfrak{E}_{\mathbb{C}})),$$

where α *is the map defined in Theorem 3.3.2.*

Proof. As in the proof of Theorem 3.4.4, we may construct a metrized vector bundle $\widetilde{E}$ on $X \times \mathbb{P}^1_{\mathbb{Z}}$ such that $i_0^*\widetilde{E} \cong E$ and $i_\infty^*\widetilde{E} \cong E' \oplus E''$ are isometries, where $i_z : X \to X \times \mathbb{P}^1_{\mathbb{Z}}, x \mapsto (x,z)$. Since we hope $\widehat{\mathrm{ch}}$ satisfies properties in Theorem 3.4.4, so

$$i_0^*\widehat{\mathrm{ch}}(\widetilde{E}) - i_\infty^*\widehat{\mathrm{ch}}(\widetilde{E}) = \alpha\left(\int_{\mathbb{P}^1(\mathbb{C})} \log|z|^2 \wedge \psi(\widehat{\mathrm{ch}}(\widetilde{E}))\right) = -\alpha(\widetilde{\mathrm{ch}}(\mathfrak{E}_{\mathbb{C}})).$$

Here we view everything as an arithmetic cycle. □

Applying Proposition 3.4.8 to the exact sequence

$$\mathfrak{T}: \quad 0 \to E \to f^*(\mathrm{Taut}_{qn,n}) \otimes \mathscr{L} \to 0,$$

we finally obtain:

Definition 3.4.9 (Arithmetic Chern Characters For Vector Bundles)**.** For a metrized vector bundle E on an arithmetic variety X defined over $\mathbb{Z}$, define its arithmetic Chern character as

$$\widehat{\mathrm{ch}}(E) := f^*\widehat{\mathrm{ch}}(\mathrm{Taut}_{qn,n}) \wedge \widehat{\mathrm{ch}}(\mathscr{L}) + \alpha(\widetilde{\mathrm{ch}}(\mathfrak{T}_{\mathbb{C}})) \in \widehat{\mathrm{CH}}^{\cdot}(X)_{\mathbb{Q}}.$$

Through these definitions, we immediately conclude that $\widehat{\mathrm{ch}}$ satisfies properties in Proposition 3.4.3, and one can prove that this definition is independent of the choices of f and $\mathscr{L}$.

Exercise 3.4.10. *The* ***Todd polynomial*** *is*

$$\mathrm{td}(x_1,\cdots,x_r) := \prod_{j=1}^{r} \frac{t_j}{1-e^{-t_j}} = 1 + \mathrm{td}_1(x_1,\cdots,x_r) + \mathrm{td}_2(x_1,\cdots,x_r) + \cdots,$$

where

$$\mathrm{td}_k(x_1,\cdots,x_r) := \textit{degree } k \textit{ part in } \prod_{j=1}^{r} \frac{t_j}{1-e^{-t_j}}; \qquad \frac{t}{1-e^{-t}} = 1 + \frac{t}{2} + \frac{t^2}{12} - \frac{t^4}{720} + \cdots$$

and $x_1,\cdots,x_r$ *are elementary symmetric polynomials in* $t_1,\cdots,t_r$*. For example, one can compute* $\mathrm{td}_1(x_1,\cdots,x_r) = \frac{x_1}{2}, \mathrm{td}_2(x_1,\cdots,x_r) = \frac{x_1^2+x_2}{12}, \cdots$.

For a rank r Hermitian vector bundle E, define its ***Todd class*** *as*

$$\mathrm{td}(E) := \mathrm{td}(c_1(E),\cdots,c_r(E)) \in \bigoplus_{p\geq 0} \mathscr{A}^{p,p}.$$

Show that:

- *If* $0 \to E' \to E \to E'' \to 0$ *is exact, then* $\mathrm{td}(E) = \mathrm{td}(E') \wedge \mathrm{td}(E'')$.

- *Try to define the* ***arithmetic Todd classes*** $\widehat{\mathrm{td}}$. *You should start with the situation of metrized line bundles, and note that we have defined*

$$\widehat{c_1}(\mathscr{L}) := \left[(\mathrm{div}(s), -[\log \|s_{\mathbb{C}}\|^2])\right]$$

in Definition 3.4.5, where s is a non-zero rational section of $\mathscr{L}$.

3.5 Generalized Heat Equations

We first introduce the heat kernels, and then use their trace formulas to introduce the Selberg zeta functions and study their analytical properties. These contents will appear in the arithmetic Riemann-Roch formula as analytic torsions, which are expressed as metrics on the determinant of cohomologies, see the next section.

Let (X,g) be a n dimensional compact orientable Riemann manifold and (E,h) be a Hermitian vector bundle on X. Then one can define a metric induced by g and h on $\Gamma(X,E)$, the linear space of smooth global sections of E, to be

$$\langle \cdot,\cdot \rangle_\Gamma : \Gamma(X,E) \times \Gamma(X,E) \to \mathbb{C}, \quad \langle s,t \rangle_\Gamma := \int_X \langle s(x), t(x) \rangle_h \cdot \mathrm{vol}_g(x).$$

Recall the **Laplace-Beltrami operator** on $\Gamma(X,E)$ is

$$\Delta_{\mathrm{Bt}}^E := \sum_{i=1}^n \left(\nabla_{e_i}^h \nabla_{e_i}^h - \nabla_{\nabla_{e_i}^g e_i}^h \right) : \Gamma(X,E) \to \Gamma(X,E),$$

where ∇^h (resp. ∇^g) is the connection induced by h (resp. g) and $\{e_i\} \subseteq TX$ is a set of local coordinates. It should be emphasized that Δ_{Bt}^E is a self-adjoint operator with respect to $\langle \cdot,\cdot \rangle_\Gamma$.

Definition 3.5.1 (Generalized Laplacians)**.** If a operator has form

$$H = -\Delta_{\mathrm{Bt}}^E + F : \Gamma(X,E) \to \Gamma(X,E),$$

where $F \in \Gamma(X, \mathrm{End}(E))$, then we call it a **generalized Laplacian**. A generalized Laplacian is said **of Laplace type**, if in addition F is self-adjoint with respect to $\langle \cdot,\cdot \rangle_\Gamma$ (this implies H is self-adjoint).

Example 3.5.2. Let $0 \le k \le \dim(X)$ be an integer. The Laplace-de Rham operator $\Delta_{\mathrm{dR}} = dd^* + d^*d$ in Remark 2.1.9 is a generalized Laplacian on $\mathscr{A}^k = \Gamma(X, \bigwedge^k T^*X)$ of Laplace type. It is obviously a self-adjoint operator, because the adjoint of d is d^*. Moreover, we have an important **Weitzenböck formula**:

$$\Delta_{\mathrm{dR}} = -\Delta_{\mathrm{Bt}}^{\bigwedge^k T^*X} + \left(\sum_{i,j,u,v} R_{ijuv} e_i^* \wedge e_j^* \wedge \iota_{e_u} \wedge \iota_{e_v} + \sum_{i,j} \mathrm{Ric}_{ij} e_i^* \wedge \iota_{e_j} \right) =: -\Delta_{\mathrm{Bt}}^{\bigwedge^k T^*X} + F,$$

where ι means contraction.

Definition 3.5.3 (Generalized Heat Equations)**.** Let X be a compact orientable Riemann manifold and E be a Hermitian vector bundle on X. The **generalized heat equation** is a PDE has form $(\frac{\partial}{\partial t}+H)u(t,x)=0$, where H is a generalized Laplacian (H only acts on the second component of u) and for each $x\in X$, $u(\cdot,x):\mathbb{R}\to E_x$.

Let $x,y\in X$ and $t>0$, suppose on $\Gamma(X,E)$ we have a generalized Laplacian H. A **heat kernel** for H is a class of linear maps $p_t(x,y):E_y\to E_x$, such that $(t,x,y)\mapsto p_t(x,y)$ is differentiable (resp. second-order differentiable) with respect to t (resp. x,y) and the partial derivatives are continuous, satisfy the following two conditions:

- $p_t(x,y)$ satisfies the heat equation: $(\frac{\partial}{\partial t}+H_x)(p_t(x,y)v)=0$ for all $v\in E_y$.
- $p_t(x,y)$ satisfies the initial condition: for any $\xi\in\Gamma(X,E)$, the limit
$$\lim_{t\to 0^+}\int_X p_t(x,y)\xi(y)\mathrm{vol}_g(y)=\xi(x)$$
converges uniformly.

Theorem 3.5.4. *Let X be a compact orientable Riemann manifold and E be a Hermitian vector bundle on X, then any generalized Laplacian H has a unique heat kernel which is smooth in t,x,y.*

For example, if we consider the trivial complex line bundle on $\mathbb{R}^n$ with the standard metric, then the heat kernel for $\Delta_{\mathrm{dR}}=-\Delta_{\mathrm{Bt}}^{\bigwedge^0 T^*\mathbb{R}^n}$ is
$$p_t(x,y)=\frac{1}{\sqrt{(4\pi t)^n}}\exp\left(-\frac{\|x-y\|^2}{4t}\right),\quad x,y\in\mathbb{R}^n,t>0.$$

The heat kernel gives the fundamental solution of the Cauchy problem of the (generalized) heat equation.

Exercise 3.5.5. *Let $\gamma,\xi\in\Gamma(X,E)$. The **Cauchy problem** is*
$$\begin{cases}\left(\dfrac{\partial}{\partial t}+H\right)u(t,x)=\gamma(x);\\ \lim\limits_{t\to 0^+}u(t,x)=\xi(x).\end{cases}$$
Verify
$$u(t,x):=\int_X p_t(x,y)\xi(y)\mathrm{vol}_g(y)+\int_0^t d\tau\int_X p_{t-\tau}(x,y)\gamma(y)\mathrm{vol}_g(y)$$
is a smooth solution of this PDE.

Theorem 3.5.4 implies that the solution in Exercise 3.5.5 is unique.

Now we use the unique smooth solution of heat equation (Exercise 3.5.5) to study the behavior of eigenvalues of a given generalized Laplacian which is of Laplace type.

Definition 3.5.6 (Heat Kernel Operators)**.** Let H be a generalized Laplacian of Laplace type. For $t>0$, define
$$e^{-tH}:\overline{\Gamma(X,E)}\longrightarrow\Gamma(X,E),\quad \xi\longmapsto\left[e^{-tH}(\xi):x\mapsto\int_X p_t(x,y)\xi(y)\mathrm{vol}_g(y)\right].$$

Note that $e^{-tH}(\xi)$ is the smooth solution of a generalized heat equation with initial condition given by ξ. If we let $t \to 0^+$, this process is equivalent to completing the target of e^{-tH}, i.e. $\Gamma(X,E)$.

Proposition 3.5.7. *For $t > 0$,*

- *e^{-tH} is a compact operator.*
- *$e^{-t_1H} \circ e^{-t_2H} = e^{-(t_1+t_2)H}$.*
- *If H is of Laplace type, then e^{-tH} is self-adjoint.*
- *There is a one-to-one correspondence $\{$eigenvalues of $H\} \leftrightarrow \{$eigenvalues of $e^{-tH}\}$, given by $\lambda \leftrightarrow Ce^{-t\lambda}$ for some constant C. This correspondence reverses the order.*

Proof. First, we show e^{-tH} is a compact operator. It is only needs to show that for any bounded set $F \subseteq \overline{\Gamma(X,E)}$ (i.e. there exists a positive number C, s.t. for any $\gamma \in F$ we have $\|\gamma\|_\Gamma \le C$), $e^{-tH}(F)$ is a sequentially compact set. By Arzela-Ascoli theorem, we should prove $e^{-tH}(F)$ is uniformly bounded and equicontinuous.

- For uniform boundedness, let $\gamma \in F$ (not necessarily continuous). Since $p_t(x,y)$ is linear and smooth in x,y, the operator norm $\|p_t(x,y)\| \le M_t$ for some M_t independent of x and y. Hence,
$$\|p_t(x,y)\gamma(y)\|_h \le M_t \cdot \|\gamma(y)\|_h.$$
So $\|e^{-tH}(\gamma)(x)\|_h \le M_t \cdot \|\gamma\|_\Gamma \cdot \mathrm{vol}_g(X) \le CM_t\mathrm{vol}_g(X)$, this bound is independent of γ and x.
- For equicontinuous, let $x_1,x_2 \in X$. If the distance $d(x_1,x_2) \to 0$, we have
$$\begin{aligned}|e^{-tH}(\gamma)(x_1) - e^{-tH}(\gamma)(x_2)| &= \left|\int_X (p_t(x_1,y) - p_t(x_2,y))\gamma(y)\mathrm{vol}_g(y)\right| \\ &\le \int_X \left\|\frac{\partial p_t(x,y)}{\partial x}\right\| \cdot d(x_1,x_2) \cdot \|\gamma(y)\|_h \cdot \mathrm{vol}_g(y) \\ &\le CM_t'\mathrm{vol}_g(X) \cdot d(x_1,x_2) \to 0.\end{aligned}$$

The equality $e^{-t_1H} \circ e^{-t_2H} = e^{-(t_1+t_2)H}$ is a consequence of the uniqueness of the solution of a Cauchy problem (Exercise 3.5.5), we leave it as an exercise.

Now, suppose H is of Laplace type, define
$$\Phi(\tau) := \langle e^{-(t-\tau)H}(\gamma_1), e^{-\tau H}(\gamma_2)\rangle_\Gamma.$$
Since Exercise 3.5.5 (take $\gamma = 0$) implies $-\frac{\partial}{\partial t}\left(e^{-tH}(\xi)\right) = He^{-tH}(\xi)$, one can compute
$$\begin{aligned}\frac{\partial \Phi}{\partial \tau} &= \left\langle \frac{\partial}{\partial \tau} e^{-(t-\tau)H}(\gamma_1), e^{-\tau H}(\gamma_2)\right\rangle_\Gamma + \left\langle e^{-(t-\tau)H}(\gamma_1), \frac{\partial}{\partial \tau} e^{-\tau H}(\gamma_2)\right\rangle_\Gamma \\ &= \left\langle -\frac{\partial}{\partial (t-\tau)} e^{-(t-\tau)H}(\gamma_1), e^{-\tau H}(\gamma_2)\right\rangle_\Gamma + \left\langle e^{-(t-\tau)H}(\gamma_1), -He^{-\tau H}(\gamma_2)\right\rangle_\Gamma \\ &= \left\langle He^{-(t-\tau)H}(\gamma_1), e^{-\tau H}(\gamma_2)\right\rangle_\Gamma - \left\langle e^{-(t-\tau)H}(\gamma_1), He^{-\tau H}(\gamma_2)\right\rangle_\Gamma \\ &= \left\langle He^{-(t-\tau)H}(\gamma_1), e^{-\tau H}(\gamma_2)\right\rangle_\Gamma - \left\langle He^{-(t-\tau)H}(\gamma_1), e^{-\tau H}(\gamma_2)\right\rangle_\Gamma \equiv 0,\end{aligned}$$

so $\Phi(\tau)$ is a constant function on $(0,t)$. Consider the continuation we have $\lim_{\tau\to 0^+}\Phi(\tau) = \lim_{\tau\to t^-}\Phi(\tau)$, i.e.

$$\langle e^{-tH}(\gamma_1),\gamma_2\rangle_\Gamma = \langle \gamma_1, e^{-tH}(\gamma_2)\rangle_\Gamma.$$

Finally, let us show the correspondence. Assume $e^{-tH}(\xi) = \eta_t\xi$ where η_t is a function only of $t \in \mathbb{R}$, by Exercise 3.5.5 (take $\gamma = 0$) we have

$$\eta_t H(\xi) = H\big(e^{-tH}(\xi)\big) = -\frac{\partial}{\partial t}\big(e^{-tH}(\xi)\big) = -\eta_t'\xi.$$

So $H\xi = (-\log\eta_t)'\xi$ and hence $\lambda := (-\log\eta_t)'$ is independent of t (this implies the constant λ is an eigenvalue of H), thus there must be $\eta_t = Ce^{-t\lambda}$, i.e. the eigenvalues of e^{-tH} give the eigenvalues of H. By spectral theorem H does not have additional eigenvalues, since eigensubspaces of e^{-tH} are eigensubspaces of H and they span the same whole space. □

By the **spectral theorem** of compact self-adjoint operators, we claim that for any $t > 0$, the spectrum (i.e. the set of eigenvalues) of e^{-tH} is a countable subset of $\mathbb{C}$ (the only possible accumulation point for this set is 0), and for each non-zero element in the spectrum, the eigensubspace of this element is finite dimensional.

Proposition 3.5.8. *If H is a generalized Laplacian of Laplace type, then the set of eigenvalues of H has a lower bound.*

Proof. Fix a $t > 0$. Since H is of Laplace type, the spectrum of e^{-tH} is contained in $\mathbb{R}$. By Proposition 3.5.7 we know e^{-tH} is a compact operator hence a bounded operator, so the spectrum of e^{-tH} is contained in $[0,D]$ for some $D \in \mathbb{R}$. This means the set of eigenvalues of H has a lower bound. □

Remark 3.5.9. With the notations in Example 3.5.2, the metric $\langle\cdot,\cdot\rangle_\Gamma$ is now the L^2-scalar product. Use the theory of heat kernel operators one can deduce the Hodge decomposition $\mathscr{A}^k = \Gamma(X, \bigwedge^k T^*X) = \ker(\Delta_{\mathrm{dR}}) \oplus \mathrm{im}(d) \oplus \mathrm{im}(d^*)$.

For a **semi-positive** generalized Laplacian H of Laplace type (for example, Δ_{dR}), by Proposition 3.5.8 we can assume the non-zero numbers in the set of eigenvalues of H are sorted by $\le$ (counted with multiplicity):

$$0 < \lambda_1 \le \lambda_2 \le \cdots.$$

Consider the theta series attached to this sequence

$$\theta_H(t) := \sum_{i\ge 1} e^{-t\lambda_i},$$

we conclude that $\theta_H(t)$ converges for $t > 0$ and induces a **Selberg zeta function**

$$\zeta_H(s) := \sum_{i\ge 1}\frac{1}{\lambda_i^s} = \frac{1}{\Gamma(s)}\int_0^\infty \theta_H(t)t^{s-1}dt.$$

This function converges for $\mathrm{Re}(s)$ sufficiently large and has a meromorphic continuation to the whole complex plane.

Formally, differentiate term by term we have

$$\zeta_H'(0) = -\sum_{i\geq 1} \log \lambda_i = -\log(\text{"determinant" of } H),$$

so we can define the **determinant** of H as $\det(H) := e^{-\zeta_H'(0)}$. It should be noted that this value is always infinite, so we need to regularize it into a finite number.

Proposition 3.5.10. *The limit*

$$\lim_{\varepsilon\to 0}\left(\int_\varepsilon^\infty \theta_H(t)\frac{dt}{t} + (\gamma_e + \log\varepsilon)\cdot\zeta_H(0)\right)$$

*exists, where γ_e is the Euler constant. This is called the **regularization** of $\zeta_H'(0)$, also denoted by $\zeta_H'(0)$.*

3.6 Metrics on the Determinant of Cohomology

3.7 Arithmetic Riemann-Roch Theorem

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