Introduction to Automorphic Representations

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CONTENTS:

Lecture 1.

Elliptic Curves, Modular Forms

· Elliptic Curves: A complex projective algebraic curve with genus 1.

Some basic facts:

$$\{\{\text{Elliptic Curves}\}\} \xleftrightarrow{1-1} \{\{\text{Complex Torus}\}\} \xleftrightarrow{1-1} SL_2(\mathbb{Z})\backslash\mathcal{H}$$

$\{\{\text{Lattices}\}\}/\sim$ (above Complex Torus)

$$\mathbb{C}(\wp, \wp'),\ \wp \text{ Weierstrass function} \longleftarrow \mathbb{C}/\Lambda,\ \Lambda = \mathbb{Z}z_1 + \mathbb{Z}z_2 \longmapsto z_1/z_2 \text{ if } \operatorname{im}(z_1/z_2) > 0.$$

<u>Def</u> (Congruence subgroups). Let $N \geq 1$.

$$\Gamma(N) := \left\{ \gamma \in SL_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}$$

$$\cap|$$

$$\Gamma_0(N) := \left\{ \gamma \in SL_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\}$$

<u>Moduli problems:</u>

· $\Gamma_0(N)\backslash\mathcal{H} \xrightarrow{1-1} \left\{ (E, C) \;\middle|\; \begin{array}{l} \cdot\ E : \text{elliptic curve}/\mathbb{C} \\ \cdot\ C \subseteq E[N] := \operatorname{Ker}[N] \cong \mathbb{Z}/N \times \mathbb{Z}/N \\ \text{a cyclic subgroup of order } N \end{array} \right\}$

$$
\begin{tikzcd}
\tau \arrow[r, mapsto] \arrow[d, "{\substack{\text{Covering map} \\ \deg = \# SL_2(\mathbb{Z})/\Gamma_0(N) \\ = N \cdot \prod_{p \mid N} \frac{1+p}{p}}}"'] & \left(E_\tau := \mathbb{C}/\Lambda_\tau,\ \langle \frac{1}{N} + \Lambda_\tau \rangle\right) \arrow[d, "{\substack{\text{Forget 'c'} \\ \deg = \#\text{ ord } N \text{ subgp inside } \mathbb{Z}/N \times \mathbb{Z}/N \\ = N \cdot \prod_{p \mid N} \frac{1+p}{p}}}"] \\
SL_2(\mathbb{Z}) \backslash \mathcal{H} \arrow[r, leftrightarrow, "1-1"] & (\!(\text{Elliptic Curves})\!) \\
\Gamma(N) \backslash \mathcal{H} \arrow[u, "\text{Covering map}"] \arrow[r, leftrightarrow] & \left\{ (E, (P,Q)) \;\middle|\; \begin{array}{l} \cdot\ E: \text{elliptic curve}/\mathbb{C} \\ \cdot\ (P,Q) \in E[N]^2 \text{ generates} \\ \ \ E[N], \text{ s.t. } e_N(P,Q) = e^{2\pi i/N} \end{array} \right\} \arrow[u, "\text{Forget '}(P,Q)\text{'}"']
\end{tikzcd}
$$

Weil pairing.

Consider a lattice $\Lambda := \mathbb{Z}z_1 + \mathbb{Z}z_2 \subseteq \mathbb{C}$.

Prop: Suppose f is a mermorphic function on $\mathbb{C}$ s.t. $\forall \lambda \in \Lambda, z \in \mathbb{C}$, $f(z+\lambda) = f(z)$, then f defines a mermorphic function on the Riemann surface $\mathbb{C}/\Lambda$.

- If $f \in \mathcal{O}(\mathbb{C})$, then f is a constant. (Liouville thm).
- Resuides: $\sum_{z \in \mathbb{C}/\Lambda} \mathrm{Res}_z(f) = 0$. $\#\mathrm{Zero}(f) = \#\mathrm{Pole}(f)$. (Since f'/f is bipreadic).

Example: Define:

$$\wp(z) := \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus 0} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right). \quad \text{even function.}$$

$\wp$ has Laurent expension in $0 < |z| < \min\{|\lambda| : 0 \neq \lambda \in \Lambda\}$:

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1)\, G_{2n+2}(\Lambda) z^{2n}.$$

where $G_k(\Lambda) := \sum_{\lambda \in \Lambda \setminus 0} \frac{1}{\lambda^k}$. it convergence absolutely when $k \geq 4$. Moreover,

$$(\wp')^2 = 4\wp^3 - 60\, G_4(\Lambda)\, \wp - 140\, G_6(\Lambda).$$

This induces a bijection:

$$(\mathbb{C}/\Lambda) \setminus \{\text{poles of } \wp\} \xleftrightarrow{1-1} \{(x,y) \in \mathbb{C}^2 \mid y^2 = 4x^3 - 60G_4 x - 140 G_6\}$$
$$z \longmapsto (\wp, \wp')$$

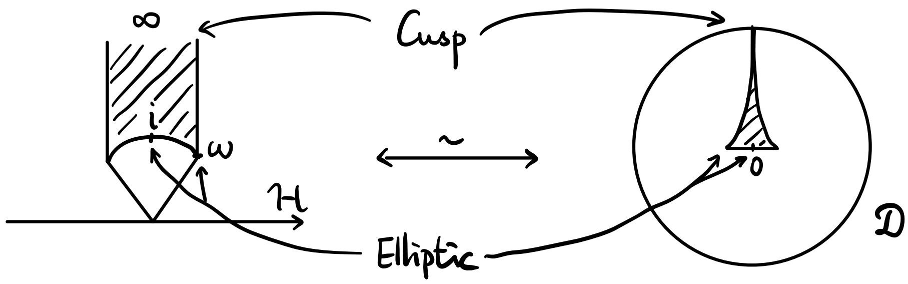
makes $\mathbb{C}/\Lambda$ a compact Riemann surface, and $\mathcal{M}_{\mathbb{C}/\Lambda} \cong \mathbb{C}(\wp, \wp')$.

Def: (Modular forms).

- The action of $SL_2(\mathbb{R})$ on the upper plane $\mathcal{H}$.

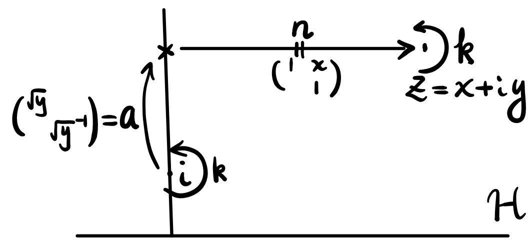
$$SL_2(\mathbb{R}) \curvearrowright \mathcal{H}: \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \longmapsto \frac{az+b}{cz+d}.$$

Fact: Every lattice in $SL_2(\mathbb{R})$ (e.g. $SL_2(\mathbb{Z})$) determines a fundamental domain.



Remark: (Iwasawa decomposition of $SL_2(\mathbb{R})$ or $PSL_2(\mathbb{R})$).

- Dynamical version: $PSL_2(\mathbb{R}) \longrightarrow T^1\mathcal{H}$, the unit sphere bundle of $\left(\mathcal{H}, \frac{|dz|^2}{\mathrm{Im}(z)^2}\right)$.
  $$g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \left(gi, \frac{i}{(ci+d)^2}\right).$$
  The goal of "ergodic theory" is to study the dynamical behaviors of geodesic flows on $\mathcal{H}$.
- Algebraical version: $SL_2(\mathbb{R}) = \underbrace{AN}_{\substack{\| \\ P=\begin{pmatrix} * & * \\ & * \end{pmatrix},}} K$, where: $A=\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$; $N=\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$; $K=SO_2(\mathbb{R})$.
  the parabolic subgroup



  - The induced measures?

- Let $k\in\mathbb{Z}$, $f:\mathcal{H}\to\mathbb{C}$ a holomorphic function, Γ a congruence subgroup. If
$$f\left(\frac{az+b}{cz+d}\right)=(cz+d)^k f(z), \quad \forall g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma, \ \forall z\in\mathcal{H},$$
and f is holomorphic at ∞, then f is a <u>level Γ, weight k modular form</u>.

- <u>Cusp form</u>: If $f(z)=\sum_{n=0}^{\infty} a_n q^n$, and $a_0=0$, then f is a cusp form.

<u>Examples</u>: (Eisenstein series). Let $k\geq 4$ be an even number. Define:
$$G_k(z) := G_k(\mathbb{Z}z+\mathbb{Z}1).$$
as a <u>Eisenstein series</u> with level $\Gamma(1)$, weight k. More precisely,
$$G_k(z) = \sum_{n\in\mathbb{Z}\setminus 0} \frac{1}{n^k} + \sum_{(c,d)\in\mathbb{Z}^2\setminus 0,\, c\neq 0} \frac{1}{(cz+d)^k}$$
$$= 2\zeta_{\mathbb{Q}}(k) + \sum_{c=1}^{\infty}\sum_{d=-\infty}^{\infty} \frac{2}{(cz+d)^k}$$
$$= 2\zeta_{\mathbb{Q}}(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{c=1}^{\infty}\sum_{n=1}^{\infty} n^{k-1} q^{cn}.$$
where $q=e^{2\pi i z}$.

Fact: (Poisson summation)
$$\sum_{n\in\mathbb{Z}} \frac{1}{(n-z)^k} = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \frac{e^{2\pi i n x}}{(x-z)^k}dx$$
$$= 2\pi i \sum_{n=1}^{\infty} \mathrm{Res}_z\left(\frac{e^{2\pi i n x}}{(x-z)^k}\right)$$
$$= (2\pi i)^k \sum_{n=1}^{\infty} n^{k-1} \frac{e^{2\pi i n z}}{(k-1)!}$$

This means $G_k(z)$ is not a cusp form. Define:

$$j := 1728 \frac{G_4^3/8\zeta_4^3}{G_4^3/8\zeta_4^3 - G_6^2/4\zeta_6^2} = q^{-1} + 744 + 196884q + \cdots.$$

Facts: The geometry of $\mathcal{H}$, and the modular curve.

- $\mathrm{Hol}(\mathcal{H}) \cong \mathrm{Iso}^+(\mathcal{H}) \cong PSL_2(\mathbb{R})$.
- Theory of covering spaces.
- Elliptic Points: $z \in \mathcal{H}$, $\mathrm{Stab}_{PSL_2(\mathbb{Z})}(z) \neq \{id\}$.
- Cusp Points: limit (rational) points in $\mathbb{R} \cup \{\infty\}$ of some fundamental domain

$$\begin{array}{ccc} \mathcal{H} & & \{1\} \\ \downarrow & & \uparrow \\ \Gamma\backslash\mathcal{H} & & \pi_1(\Gamma\backslash\mathcal{H}) \cong \mathrm{Galois} \end{array}$$

Example:
- $\Gamma(1) = SL_2(\mathbb{Z})$. Elliptic pts: ω, i; Cusp pts: ∞. (up to orbits)
- the number of elliptic points of $\Gamma(N) = \begin{cases} 0, & N \neq 1. \\ 2; & N = 1. \end{cases}$

  the number of cusp points of $\Gamma(N) = \begin{cases} \frac{N^2}{2} \prod_{p \mid N} (1-p^{-2}), & N \neq 2. \\ 3, & N = 2. \end{cases}$

Theorem: (Modular curves and modular forms).

Let Γ be a congruence subgroup. Then:

- The fundamental domain $\Gamma\backslash\mathcal{H}$ is a non-compact Riemann surface. The complex structure near $z \in \mathcal{H}$ is $\omega \mapsto \omega^{|\mathrm{stab}_\Gamma(z)|}$. Its compactification is the so-called "modular curve", denoted by $X(\Gamma)$.

  e.g. $X(1) := X(\Gamma(1)) = \mathbb{P}^1(\mathbb{C})$.

Cusp forms $S_k(\Gamma)$

- (The katz sheaf). The space of holomorphic modular forms $M_k(\Gamma)$ is the global section of a sheaf $\mathcal{O}_{X(\Gamma)}(\pi_\Gamma^*(\infty)^{\otimes k/12})$ on $X(\Gamma)$, where $\pi_\Gamma : X(\Gamma) \to X(1)$ is the mermorphic covering map. (weight $12 \mid k$)

  More generally, define a line bundle ω_k on $X(\Gamma)$:

$$\forall\, \mathcal{U} \subseteq X(\Gamma) \text{ open},\ \omega_k(\mathcal{U}) := \left\{ \begin{array}{l} f \in \mathcal{O}((\mathcal{H} \to \Gamma\backslash\mathcal{H})^{-1}(\mathcal{U})), \text{ s.t.} \\ f(\gamma z) = (cz+d)^k f(z),\ \forall \gamma \in \Gamma. \end{array} \right\},$$

  then $H^0(X(\Gamma), \omega_k) \cong M_k(\Gamma)$. In particular, $\omega_2 \cong \Omega_{X(\Gamma)}(\mathrm{Cusp})$.

Consider $\Gamma = SL_2(\mathbb{Z})$, $M_k(1) := M_k(\Gamma)$, $S_k(1) := S_k(\Gamma)$.

- $\bigoplus_{k=0}^{\infty} M_k(1) \cong \mathbb{C}[G_4, G_6]$. In particular, $M_k(1) = \mathrm{span}_{\mathbb{C}}\{G_4^m G_6^n\}$, $4m + 6n = k$.

and $\dim_{\mathbb{C}} M_k(1) = \begin{cases} 0 & k<0, \text{ or } k \text{ odd} \\ [k/12] & k\equiv 2 \pmod{12}, k\geq 0 \\ [k/12]+1 & k\not\equiv 2 \pmod{12}, k\geq 0 \end{cases}$ (Riemann-Roch).

- $M_{k-12}(1) \xrightarrow{\sim} S_k(1)$, $f \mapsto f\cdot\Delta$, where $\Delta = (60G_4)^3 - 27(140G_6)^2 \in S_{12}(1)$.

$\rightsquigarrow$ Compute $\dim_{\mathbb{C}} S_k(1)$.

Lecture 2.

$\mathbb{Q}$-Tate Thesis and the GL_1-Langlands

Recall: (Ring of adèles of $\mathbb{Q}$).

$$\mathbb{A} := \prod_{p\leq\infty}{}'^{\mathbb{Z}_p} \mathbb{Q}_p \quad \text{with the "restrict product topology".}$$

Facts:
- $\mathbb{A}$ is a LCG (locally compact group).
- $\mathbb{A}$ is a topological ring with unit group $\mathbb{A}^\times$, the idèle group of $\mathbb{Q}$.
- $\mathbb{Q}$ is a discrete closed subgroup of $\mathbb{A}$.
- $\mathbb{A}/\mathbb{Q}$ is compact. In particular, $\mathbb{A}/\mathbb{Q} \cong \hat{\mathbb{Z}} \times (\mathbb{R}/\mathbb{Z})$.
- (Strong approximation): let v_0 be a place of $\mathbb{Q}$, then $\mathbb{Q}$ is dense in
$$\mathbb{A}_{v_0} := \prod_{p\neq v_0}{}'^{\mathbb{Z}_p} \mathbb{Q}_p$$
In particular, $\mathbb{Q}$ is dense in $\mathbb{A}_\infty$ ($\mathbb{A}_f$).

Definition: (Idèle class group). If K is a number field, let
$$\mathbb{A}_K := \mathbb{A} \otimes_{\mathbb{Q}} K.$$
Define $\mathcal{C}\ell(K) := \mathbb{A}_K^\times / K^\times$ to be the idèle class group of K.

There is a (continuous) homomorphism $|\cdot| : \mathbb{A}_K^\times \longrightarrow \mathbb{R}_{>0}$, $(x_v) \longmapsto \prod_v |x_v|_v$, called the absolute value on $\mathbb{A}_K^\times$.

Remark:
- We have isomorphisms:
$$\mathcal{C}\ell(K) / \text{maximal compact subgroup}_{(\text{fin})} \times K^\times_{\infty(\text{inf})} \simeq CH^1(X) \simeq \mathrm{Pic}(\mathrm{Spec}\, \mathcal{O}_K) \simeq \mathrm{Cl}(\mathcal{O}_K).$$
- Class Field theory:

The Artin reciprocity law: $\mathcal{C}\ell(\mathbb{Q}) / \mathrm{Norm}_{K/\mathbb{Q}}(\mathcal{C}\ell(K)) \xrightarrow{\sim} \mathrm{Gal}(K/\mathbb{Q})$.

where $K/\mathbb{Q}$ is a finite abelian extension. In particular, for ideal class groups, we have: $\mathrm{Cl}(\mathcal{O}_K) \xrightarrow{\sim} \mathrm{Gal}(K/\mathbb{Q})$, $I = \prod_{\mathfrak{p}} \mathfrak{p}^{e_{\mathfrak{p}}} \longmapsto \prod_{\mathfrak{p}} \left(\frac{K/\mathbb{Q}}{\mathfrak{p}}\right)^{e_{\mathfrak{p}}}$.

where $\left(\frac{\cdot}{p}\right)$ is Hilbert symbol, the RHS "$\prod$" means composite.

- Product formula: $\forall x \in \mathbb{Q}^\times \subseteq \mathbb{A}^\times$, $|x|=1$.
- $Cl(\mathbb{Q}) = \mathbb{A}^\times/\mathbb{Q}^\times$ is not compact. However, if write $\mathbb{A}^1 := \mathrm{Ker}(\mathbb{A}^\times \xrightarrow{|\cdot|} \mathbb{R}_{>0})$, then $Cl^1(\mathbb{Q}) := \mathbb{A}^1/\mathbb{Q}^\times$ is compact. and:
  - $\mathbb{A}^\times \xrightarrow{\sim} \mathbb{A}^1 \times \mathbb{R}_{>0}$. $\vec{x} = (x_p) \longmapsto \left(\left((x_p), \frac{x_\infty}{|\vec{x}|}\right), |\vec{x}|\right)$.
  - $Cl^1(\mathbb{Q}) \xrightarrow{\sim} \hat{\mathbb{Z}}^\times$. $((x_p),1)\mathbb{Q}^\times \longleftarrow (x_p)$.
  - $Cl(\mathbb{Q}) \xrightarrow{\sim} Cl^1(\mathbb{Q}) \times \mathbb{R}_{>0}$, $(x_p)\mathbb{Q}^\times \longmapsto \left(\left((x_p), \frac{x_\infty}{|(x_p)|}\right)\mathbb{Q}^\times, |(x_p)|\right)$.

Theorem: (Tate, Pontryagin duality of local fields). Let $K = \mathbb{Q}_p, \mathbb{R}, \mathbb{C}$. then there is an isomorphism of topological abelian groups:

$K \xrightarrow{\sim} \hat{K}$, $s \mapsto \left(\chi_s : K \to S^1, t \mapsto \tilde{\chi}(st)\right)$. where $\hat{K} := \mathrm{Hom}_{cts}(K, S^1)$ is the group of unitary characters, $\tilde{\chi}$ is a "priori" non-trivial character.

We take:
- $K = \mathbb{R}$, $\tilde{\chi}: \mathbb{R} \to S^1$, $x \mapsto e^{-2\pi i x}$.
- $K = \mathbb{C}$. $\tilde{\chi}: \mathbb{C} \to S^1$. $x \mapsto e^{-2\pi i(x+\bar{x})}$
- $K = \mathbb{Q}_p$, $\tilde{\chi}: \mathbb{Q}_p \to S^1$. $\sum_{-N}^{\infty} a_j p^j \mapsto \exp\left(2\pi i \sum_{-N}^{-1} a_j p^j\right)$.
  $\tilde{\chi}(x) = 1$ if and only if $x \in \mathbb{Z}_p$.
- (K = extension of above) $\tilde{\chi}' := \tilde{\chi} \circ \mathrm{trace}$.

Haar measures:
Real & Complex analysis
- $K = \mathbb{R}$. dx the Lebesgue measure. $d^\times x := \frac{dx}{|x|}$ on $\mathbb{R}^\times$.
- $K = \mathbb{C}$. $dz = d(x+iy) := 2dxdy$. $d^\times z := \frac{dz}{z\bar{z}}$ on $\mathbb{C}^\times$.
- $K = \mathbb{Q}_p$, select dx to make $\int_{\mathbb{Z}_p} dx = 1$. $d^\times x := \frac{1}{1-p^{-1}} \frac{dx}{|x|}$.

Proposition: On $\mathbb{Q}_p$ we have:

1) (Change Variable) $\forall$ measurable $X \subseteq \mathbb{Q}_p$, $\forall a \in \mathbb{Q}_p$. $\int_{aX} dx = |a|_p \int_X dx$. So: $\int_{aX} f(y)dy = |a|_p \int_X f(ay)dy$. In particular, $\int_{a+p^n\mathbb{Z}_p} dx = p^{-n}$.
2) (Multiplicative Group) $\int_{\mathbb{Z}_p^\times} d^\times x = \int_{\mathbb{Z}_p} dx = 1$; $\int_{\mathbb{Z}_p \setminus 0} |x|^t d^\times x = \frac{1}{1-p^{-t}}$ ($t \in \mathbb{C}$, $Re(t) > 0$).
3) (Characteristic Function). The Fourier transform of $\mathbb{1}_{a+(p)^n}$ is
$$\widehat{\mathbb{1}_{a+(p)^n}}(s) := \widehat{\mathbb{1}_{a+(p)^n}}(\chi_s) = \overline{\chi_s(a)}\, p^{-n} \cdot \mathbb{1}_{(p)^{-n}}.$$
4) (Duality). Write $\mathbb{Z}_p^\perp := \{\chi \in \hat{\mathbb{Z}}_p : \chi(\mathbb{Z}_p) = 1\}$. then with the dual measure $d\chi$ on $\hat{\mathbb{Z}}_p$, $\int_{\mathbb{Z}_p^\perp} d\chi = 1$.

Proof: Only prove 2) & 3):

Suppose $G \in LCG$ abelian with Haar measure μ. Then $\hat{G} := \mathrm{Hom}_{cts}(G, S^1)$ has a unique Haar measure (the dual measure) s.t. $\forall f \in \mathcal{C}_c(G)$, $\|f\|_2 = \|\hat{f}\|_2$.

2). $\int_{\mathbb{Z}_p^\times} d^\times x = \frac{1}{1-p^{-1}} \int_{\mathbb{Z}_p^\times} \frac{dx}{|x|}$

$(|x|=1) = \frac{1}{1-p^{-1}} \int_{\mathbb{Z}_p \setminus (p)} dx$

$$= \frac{1}{1-p^{-1}} \left(\int_{\mathbb{Z}_p} dx - \int_{(p)\mathbb{Z}_p} dx \right) \overset{1)}{=} \frac{1}{1-p^{-1}} (1-p^{-1}) = 1.$$

Moreover, note that 1) implies $\int_{p^k\mathbb{Z}_p} dx = p^{-k}$, so:

$$\int_{\mathbb{Z}_p \setminus 0} |x|^t d^\times x = \sum_{k=0}^{\infty} \int_{p^k\mathbb{Z}_p \setminus p^{k+1}\mathbb{Z}_p} \frac{|x|^{t-1}}{1-p^{-1}} dx = \frac{1}{1-p^{-1}} \sum_{k=0}^{\infty} p^{-k(t-1)} \left(\int_{p^k\mathbb{Z}_p} dx - \int_{p^{k+1}\mathbb{Z}_p} dx \right) = \frac{1}{1-p^{-t}}.$$

3). $\widehat{\mathbb{1}_{a+(p)^n}}(s) = \int_{a+(p)^n} \overline{\chi_s(y)}\, dy = \int_{(p)^n} \overline{\chi_s(a+y)}\, dy = \overline{\chi_s(a)} \int_{(p)^n} \overline{\chi_s(y)}\, dy.$

Note that $y \in \mathbb{Z}_p$ implies $p^n s y \in s(p)^n$, and $s(p)^n \subseteq \mathbb{Z}_p$ then $\tilde{\chi}$ is trivial.

Hence, $\int_{(p)^n} \overline{\chi_s(y)}\, dy = \int_{(p)^n} \overline{\tilde{\chi}(sy)}\, dy \overset{1)}{=} p^{-n} \int_{\mathbb{Z}_p} \overline{\tilde{\chi}(p^n s y)}\, dy = p^{-n} \int_{\mathbb{Z}_p} dy = p^{-n}.$ $\square$

Theorem: (Tate, Classification of quasi-characters).

$\forall\ \chi : K^\times \to \mathbb{C}^\times$, $y \mapsto \chi(y)$ continuous homomorphism, we have

$$\chi(y) = \chi_0(y_U) \cdot |y|^s, \text{ where } \chi_0 \in \widehat{U_K},\ U_K = \begin{cases} \{\pm 1\}, & K = \mathbb{R} \\ S^1, & K = \mathbb{C} \\ \mathbb{Z}_p^\times, & K = \mathbb{Q}_p \end{cases},\ s \in \mathbb{C} \text{ depend on } \chi.$$

$y_U \in \mathrm{Ker}(|\cdot|)$

$\sigma(\chi) := \mathrm{Re}(s)$.

i.e. quasi-character $\underset{}{\overset{\text{uniquely}}{=\!=}}$ ramified (unitary character) $\times$ unramified.

Definition: Let $f \in \mathcal{S}(K)$ be a schwartz function. define the local ζ function as:

If K is p-adic, then f is locally const. & with compact support.

$$\zeta(f, \cdot) : \{\chi \text{ quasi-character}, \sigma(\chi) > 0\} \longrightarrow \mathbb{C},$$
$$\zeta(f, \chi) := \int_{K^\times} f(x)\chi(x) d^\times x.$$

Facts: · Well-defined!

· $\forall f, g \in \mathcal{S}(K)$, $\forall \chi$ s.t. $0 < \sigma(\chi) < 1$, it is easy to verify:

$$\zeta(f, \chi) \cdot \zeta(\hat{g}, |\cdot|\chi^{-1}) = \zeta(\hat{f}, |\cdot|\chi^{-1}) \cdot \zeta(g, \chi).$$

$\rightsquigarrow \rho(\chi) := \frac{\zeta(f, \chi)}{\zeta(\hat{f}, \hat{\chi})}$ does not depend on f. ($\hat{\chi} := |\cdot|\chi^{-1}$).

THEOREM: (TATE, LOCAL THEORY).

$\forall f \in \mathcal{S}(K)$, $\zeta(f, \chi)$ defines a mermomorphic function on the Riemann surface $X(K^\times) := \{\text{quasi-characters of } K^\times\}$, $\chi|\cdot|^s \longmapsto s \in \mathbb{C}$.

and $\rho(\chi)$ is also a mermomorphic function. More precisely:

- $K=\mathbb{R}$, $X(K^\times)=\{|\cdot|^s : s\in\mathbb{C}\}\sqcup\{\mathrm{sgn}(\cdot)|\cdot|^s : s\in\mathbb{C}\}$,
$$\rho(|\cdot|^s)=2^{1-s}\pi^{-s}\cos\left(\tfrac{\pi s}{2}\right)\Gamma(s),\quad \rho(\mathrm{sgn}(\cdot)|\cdot|^s)=-i\,2^{1-s}\pi^{-s}\sin\left(\tfrac{\pi s}{2}\right)\Gamma(s).$$
- $K=\mathbb{C}$, $X(K^\times)=\bigsqcup_{n\in\mathbb{Z}}\{\chi_n|\cdot|^s : s\in\mathbb{C}\}$, where $\chi_n:\mathbb{C}^\times\to S^1$, $re^{it}\mapsto e^{itn}$.
$$\rho(\chi_n|\cdot|^s)=(-i)^{|n|}(2\pi)^{1-2s}\frac{\Gamma(s+\frac{|n|}{2})}{\Gamma(1-s+\frac{|n|}{2})}.$$
- $K=\mathbb{Q}_p$, write $\chi=\chi_0|\cdot|^s$.
  - χ_0 trivial, $\rho(|\cdot|^s)=\dfrac{1-p^{s-1}}{1-p^{-s}}$.
  - χ_0 non-trivial, exercise!

<u>Proof</u>: Only prove the case $K=\mathbb{C}$. Consider $\chi_{-n}|\cdot|^s$, $s\in\mathbb{C}$. Take $f_{-n}(z):=z^n e^{-\pi z\bar{z}}$.
$$\begin{aligned}\zeta(f_{-n},\chi_{-n}|\cdot|^s)&=\int_{\mathbb{C}^\times}z^n e^{-\pi z\bar z}\chi_{-n}(z)(z\bar z)^s d^\times z \quad (|z|:=z\bar z)\\ &=\int_{\theta=0}^{2\pi}\int_{r=0}^{+\infty}e^{-\pi r^2}r^{2s+n}\frac{2r\,dr\,d\theta}{r^2}\quad (z\to re^{i\theta},\ d^\times z=dz/r^2)\\ &=4\pi\int_0^{+\infty}e^{-\pi r^2}r^{2s+n-1}dr\quad (\pi r^2\to t)\\ &=\underline{2\pi^{1-s-\frac{n}{2}}\Gamma(s+\tfrac{n}{2})}.\end{aligned}$$
One needs to compute $\zeta(\widehat{f_{-n}},\widehat{\chi_{-n}|\cdot|^s})$. Let $g(z):=e^{-\pi z\bar z}=z^{-n}f_{-n}(z)$, its Fourier transformation is:
$$\begin{aligned}\hat g(z):=\hat g(\chi_z)&=\int_{\mathbb{C}}e^{-\pi w\bar w}e^{2\pi i(zw+\bar z\bar w)}dw\quad \begin{pmatrix}z=x+iy\\ w=u+iv\\ dw=2du\,dv\end{pmatrix}\\ &=2\int_{\mathbb{R}}\int_{\mathbb{R}}e^{-\pi(u^2+v^2)}e^{4\pi i(ux-vy)}du\,dv\\ &=2e^{-4(x^2+y^2)}\underbrace{\int_{\mathbb{R}}e^{-\pi(u-2ix)^2}du\int_{\mathbb{R}}e^{-\pi(v+2iy)^2}dv}_{=1}\\ &=2e^{-4\pi z\bar z}.\end{aligned}$$
Hence, $\frac{\partial^n}{\partial z^n}(2e^{-4\pi z\bar z})=2(-4\pi\bar z)^n e^{-4\pi z\bar z}=(2\pi i)^n\int_{\mathbb{C}}w^n e^{-\pi w\bar w}e^{2\pi i(zw+\bar z\bar w)}dw$
$\Rightarrow \widehat{f_{-n}}(z)=2f_{-n}(2i\bar z)$. So: $\qquad = (2\pi i)^n\widehat{f_{-n}}(z)$.
$$\zeta(\widehat{f_{-n}},\widehat{\chi_{-n}|\cdot|^s})=\zeta(2f_{-n}(2i\bar z),\chi_n|\cdot|^{1-s})=\underline{i^n 2^{2s}\pi^{s-\frac{n}{2}}\Gamma(1+\tfrac{n}{2}-s)}.\ (n\geq 0).\quad\square$$

<u>THEOREM</u>: (TATE, GLOBAL THEORY).

- $\int_{\mathbb{A}/\mathbb{Q}}d\vec{x}=1$ (with the product measure).
- One has: $\Theta:\mathbb{A}\xrightarrow{\sim}\hat{\mathbb{A}}:=\prod'^{\mathbb{Z}_p^\perp}_{p\leq\infty}\hat{\mathbb{Q}}_p$, given by $(x_p)\longmapsto\begin{pmatrix}\mathbb{A}\longrightarrow S^1\\ (y_p)\longmapsto\prod_p\tilde\chi_p(x_py_p)\end{pmatrix}$.
- One has: $\hat{\mathbb{Q}}\cong\mathbb{A}/\mathbb{Q}$. (Exercise!)

- Define $\mathcal{S}(\mathbb{A}) := \{ \sum_i^{\infty} a_i f_i : a_i \in \mathbb{C}, f_i = \bigotimes_p f_{i,p}, f_{i,p} \in \mathcal{S}(\mathbb{Q}_p), \text{ and } f_{i,p} = \mathbb{1}_{\mathbb{Z}_p} \text{ a.e. } p \}$ as the space of schwartz functions. then:
  - The Fourier transform of $f \in \mathcal{S}(\mathbb{A})$ is

$$\hat{f}(\vec{y}) := \int_{\mathbb{A}} f(\vec{x}) \overline{\Theta(\vec{y})(\vec{x})} d\vec{x} \in \mathcal{S}(\hat{\mathbb{A}}) \cong \mathcal{S}(\mathbb{A}).$$

    Hence, $\hat{\hat{f}} = f$.
  - (Riemann-Roch) $\forall \vec{x} \in \mathbb{A}^{\times}$, $f \in \mathcal{S}(\mathbb{A})$, we have:

$$\sum_{y \in \mathbb{Q}} f(y\vec{x}) = \frac{1}{|\vec{x}|} \sum_{y \in \mathbb{Q}} \hat{f}(y\vec{x}^{-1}).$$

    This is a special case of the "Poisson summation formula".
- (Hecke characters). $\mathbb{X} := \{ \vec{\chi} : Cl(\mathbb{Q}) \to \mathbb{C}^{\times} \text{ continuous homomorphism} \}$.
  use the isomorphism:

$$Cl(\mathbb{Q}) \xrightarrow{\sim} Cl^1(\mathbb{Q}) \times \mathbb{R}_{>0}, \quad (x_p)\mathbb{Q}^{\times} \longmapsto \left(\left((x_p), \frac{x_\infty}{|(x_p)|} \right) \mathbb{Q}^{\times}, |(x_p)| \right).$$

$$\vec{x} \longmapsto (\vec{x}_U, \vec{x}_V := |\vec{x}|).$$

  one can classify: $\forall \vec{\chi} \in \mathbb{X}$, $\vec{\chi}$ has form $\vec{\chi}(\vec{y}) = \underset{\text{ramified}}{\vec{\chi}_0(\vec{y}_U)} \cdot \underset{\text{unramified}}{|\vec{y}|^s}$. where:
  $\vec{\chi}_0 \in \widehat{Cl^1(\mathbb{Q})}$, $s \in \mathbb{C}$. $\sigma(\vec{\chi}) := \mathrm{Re}(s)$.

Compact!

- (Global ζ functions). $\forall f \in \mathcal{S}(\mathbb{A})$, define:

$$\zeta(f, \cdot) : \{ \vec{\chi} \in \mathbb{X}, \sigma(\vec{\chi}) > 1 \} \longrightarrow \mathbb{C},$$

the product measure of $d^{\times}x_p$.

$$\zeta(f, \vec{\chi}) := \int_{\mathbb{A}^{\times}} f(\vec{x}) \vec{\chi}(\vec{x}) d^{\times}\vec{x}.$$

  - Well-defined!
  - (Functional Equation): $\zeta(f, \vec{\chi}) = \zeta(\hat{f}, |\cdot|\vec{\chi}^{-1})$. $\forall f \in \mathcal{S}(\mathbb{A})$.

<u>Corollary</u>: (Functional Equation of Riemann-zeta function). $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$.

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s).$$

<u>Proof</u>: Take $f \in \mathcal{S}(\mathbb{A})$: $f = \prod_p f_p$, where $f_p = \begin{cases} e^{-\pi x^2}, & \mathbb{R} \\ \mathbb{1}_{\mathbb{Z}_p}. & \mathbb{Q}_p. \end{cases}$

Then $\zeta(f_p, |\cdot|^s) = \begin{cases} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right), & \mathbb{R} \\ \frac{1}{1-p^{-s}}, & \mathbb{Q}_p \end{cases}$ (by the proposition 2) above).

and $\zeta(\hat{f}_p, |\cdot|^{1-s}) = \begin{cases} \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right), & \mathbb{R} \\ \frac{1}{1-p^{s-1}}, & \mathbb{Q}_p \end{cases}$

$\Rightarrow$ By Theorem of Tate,

$$\begin{array}{c}\zeta(f,|\cdot|^s) = \prod_p \zeta(f_p,|\cdot|^s) = \pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\zeta(s), \\ \| \\ \zeta(\hat{f},|\hat{\cdot}|^s) = \prod_p \zeta(\hat{f}_p,|\cdot|^{1-s}) = \pi^{-\frac{1-s}{2}}\Gamma(\tfrac{1-s}{2})\zeta(1-s).\end{array} \qquad \square$$

<u>Remark</u>: (GL_1-Langlands). Let $K/\mathbb{Q}$ be an abelian extension. Then:

$$\{\text{(unitary) Galois repn of } Gal(K/\mathbb{Q})\} \overset{CFT}{\longleftrightarrow} \{\text{(unitary) repn of } Cl(\mathbb{Q})/_{Nm(Cl(K))}\}$$

So:
$$\begin{tikzcd} Gal(\mathbb{Q}^{ab}/\mathbb{Q}) \arrow[d, "proj"'] \arrow[dr, dashed] & \\ Gal(K/\mathbb{Q}) \arrow[r] & S^1 \end{tikzcd}$$

induces:
$$\begin{tikzcd} \widehat{Gal(\mathbb{Q}^{ab}/\mathbb{Q})} \arrow[r, "Artin"] & \widehat{Cl(\mathbb{Q})} \arrow[r, hook] & \mathbb{X} \\ \widehat{Gal(K/\mathbb{Q})} \arrow[u] \arrow[r, dash, "\sim"] & (Cl(\mathbb{Q})/_N)^\wedge \arrow[u] & \text{Hecke characters} \arrow[u] \arrow[d, squiggly] \\ & & \text{1-dim auto. forms} \end{tikzcd}$$

(has non-abel information)

Recall: $Cl(\mathbb{Q})/_N = \mathbb{Q}^\times \backslash \mathbb{A}^\times/_N \rightsquigarrow$ shimura varieties!

Since $GL_1(\mathbb{A}) = \mathbb{A}^\times$, one needs to study Hecke characters!

Lecture 3.

GL_2 Automorphic Forms & Representations.

From modular forms to GL_2-modular forms.

Let $f: \mathcal{H} \to \mathbb{C}$ be a modular form of weight k, level $\Gamma \subseteq SL_2(\mathbb{Z})$.

(i.e. f equivarent w.r.t. left Γ-action).

Our goal is to construct:

$$F_f: GL_2(\mathbb{R})^+ \longrightarrow \mathbb{C} \qquad \text{(automorphic forms)}.$$
$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (f|_{[g]_k})(i) = (\det g)^{\frac{k}{2}}(ci+d)^{-k} f\left(\frac{ai+b}{ci+d}\right).$$

<u>Claims</u>:

- $\forall \gamma \in \Gamma$, $F_f(\gamma g) = (f|_{[\gamma g]_k})(i) = (f|_{[g]_k})(i) = F_f(g)$.
  (i.e. invariant w.r.t. left Γ-action).
- $\forall \kappa = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in K := SO_2(\mathbb{R})$, $F_f(g\kappa) = e^{-ik\theta}F_f(g)$.
  (i.e. equivarent w.r.t. right K-action).
- $\forall \delta = \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} \in Z := Z(GL_2(\mathbb{R})^+)$, $F_f(g\delta) = \omega(\delta)F_f(g)$, where $\omega(\delta) = \begin{cases} 1, & \lambda > 0 \\ (-1)^k, & \lambda < 0 \end{cases}$
  (i.e. equivarent w.r.t. right Z-action).

<u>Fact</u>: $$\Gamma \backslash GL_2(\mathbb{R})^+ / Z\cdot K \overset{\sim}{\longrightarrow} \Gamma \backslash \mathcal{H}.$$

Proof: $GL_2(\mathbb{R})^+ \twoheadrightarrow \mathcal{H}$, $g \mapsto gi$, Kernel $= Z \cdot K$. $\square$

Study Hermitian Geometry: Some Lie Theory:

Consider $C^\infty(GL_2(\mathbb{R})^+, \mathbb{C})$, $\mathfrak{g} := \mathfrak{gl}_2(\mathbb{R})$.

- Regular Representations: $\forall g \in GL_2(\mathbb{R})^+$, $g: C^\infty(GL_2(\mathbb{R})^+, \mathbb{C}) \to C^\infty(GL_2(\mathbb{R})^+, \mathbb{C})$

$$F(x) \longmapsto F(xg).$$

$$\Downarrow$$

$\forall X \in \mathfrak{g}$, $X: C^\infty(GL_2(\mathbb{R})^+, \mathbb{C}) \to C^\infty(GL_2(\mathbb{R})^+, \mathbb{C})$

$$F(x) \longmapsto \frac{d}{dt} F(x \cdot \exp(tX))\Big|_{t=0}.$$

- Universal Enveloping Algebra: $\mathfrak{g}_\mathbb{C} := \mathfrak{g} \otimes_\mathbb{R} \mathbb{C} = \mathrm{Mat}_2(\mathbb{C})$.

Write: $z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $r = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$, $l = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$, $h = \begin{pmatrix} & -i \\ i & \end{pmatrix} \in \mathfrak{g}_\mathbb{C}$. as basis.

$$\mathcal{U}(\mathfrak{g}_\mathbb{C}) \cong \mathbb{C}[z, r, l, h] / \langle zr - rz, rl - lr - h, \cdots \rangle.$$

Then:

$Z(\mathcal{U}(\mathfrak{g}_\mathbb{C})) = \mathbb{C}[z, \Delta]$, where $\Delta = -\frac{1}{4}(h^2 + 2rl + 2lr)$ is the Casimir element.

"Laplace operator"

Definition: (Automorphic Forms, Geometric Version)

- (Z-finite): $F \in C^\infty(GL_2(\mathbb{R})^+, \mathbb{C})$ is Z-finite if $\{\Delta^n F\}_{n \geq 0}$ spans a finite dim $\mathbb{C}$-linear space.
- (K-finite): $F \in C^\infty(GL_2(\mathbb{R})^+, \mathbb{C})$ is K-finite if $\{\kappa F\}_{\kappa \in K}$ spans a finite dim $\mathbb{C}$-linear space. where $(\kappa F)(g) := F(g\kappa)$.

Suppose $\Gamma \subseteq SL_2(\mathbb{R})$. Let ω be a character of Z (equivalent to $\omega: \mathbb{R}^\times \to S^1$).

In modular forms	
	$\mathcal{A}(\Gamma \backslash GL_2(\mathbb{R})^+, \omega) :=$ space of smooth functions $F: GL_2(\mathbb{R})^+ \to \mathbb{C}$, s.t.
level structure	· (left Γ-invariant): $\forall \gamma \in \Gamma$, $F(\gamma g) = F(g)$.
twist	· (Z-invariant): $\forall \delta \in Z$, $F(g\delta) = \omega(\delta) F(g)$.
holomorphic condition "weight"	· (Finitness): F is Z-finite & K-finite.
growth condition	· (Moderate Growth): $\exists\, C, N > 0$, s.t. $\lvert F(g) \rvert < C \lVert g \rVert^N$.

Automorphic Forms on $GL_2(\mathbb{R})^+$ w.r.t. the central character ω.

arbitrary norm, e.g.: the length of $(g, \det g^{-1}) \in \mathbb{R}^5$.

- (Cusp Forms): Suppose Γ contains a $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$, $r > 0$. $F \in \mathcal{A}(\Gamma \backslash GL_2(\mathbb{R})^+, \omega)$ is cuspidal if $\forall g \in GL_2(\mathbb{R})^+$, $\int_0^r F(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} g)\, dt = 0$.
  - Not depend on the choice of r.
  - $\forall$ Cusp C, choose $P \in SL_2(\mathbb{R})$ s.t. $P(\infty) = C$. the "cuspidal at C" is defined by conjugations.

$\rightsquigarrow \mathcal{A}_0(\Gamma\backslash GL_2(\mathbb{R})^+, \omega)$, the space of cusp forms.

<u>Theorem</u>: $\Gamma \subseteq SL_2(\mathbb{Z})$ congruence subgroup.

- $f \in M_k(\Gamma) \Rightarrow F_f \in \mathcal{A}\left(\Gamma\backslash GL_2(\mathbb{R})^+, \omega: \begin{pmatrix}\lambda & \\ & \lambda\end{pmatrix} \mapsto \begin{cases} 1, & \lambda>0 \\ (-1)^k, & \lambda<0\end{cases}\right)$.
- $f \in S_k(\Gamma) \Rightarrow F_f \in \mathcal{A}_0$.

<u>Proof</u>: · (left Γ-inv.) & (Z-inv.) are proved in the claim above.

· (Finiteness). <u>Remark</u>: One needs an other coordinate on $GL_2(\mathbb{R})^+$.

$$\forall g \in GL_2(\mathbb{R})^+,\ g = \underbrace{\begin{pmatrix}\lambda & \\ & \lambda\end{pmatrix}}_{\delta} \underbrace{\begin{pmatrix} y^{\frac12} & y^{-\frac12}x \\ & y^{-\frac12}\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}}_{AN\cdot K},\quad \lambda>0,\ x\in\mathbb{R},\ y>0,\ \theta\in[0,2\pi).$$

$\rightsquigarrow$ coordinate (λ, x, y, θ). $\rightsquigarrow F_f(g) = \omega(\delta)e^{-ik\theta}y^{\frac{k}{2}}f(x+iy)$, where $gi = x+iy$.

<u>Claim</u>: $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) - y\frac{\partial^2}{\partial x\partial\theta}$ is the Laplace operator on $T^1\mathcal{H}$.

<u>Pf</u>: Consider $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = -ih$. Assume $F: GL_2(\mathbb{R})^+ \to \mathbb{C}$.

$$\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}F\right)(g) = \frac{d}{dt}\left(F\left(g\exp\left(t\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\right)\right)\right)\Big|_{t=0}$$

$$= \frac{d}{dt}F\left(\begin{pmatrix}\lambda & \\ & \lambda\end{pmatrix}\begin{pmatrix} y^{\frac12} & y^{-\frac12}x \\ & y^{-\frac12}\end{pmatrix}\begin{pmatrix}\cos(\theta+t) & -\sin(\theta+t) \\ \sin(\theta+t) & \cos(\theta+t)\end{pmatrix}\right)\Big|_{t=0}$$

$$= \frac{\partial}{\partial\theta}F(g) \Rightarrow h = i\frac{\partial}{\partial\theta}.$$

Similarly, $\ell = e^{2i\theta}\left(-iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2i}\frac{\partial}{\partial\theta}\right)$, $r = e^{-2i\theta}\left(iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{1}{2i}\frac{\partial}{\partial\theta}\right)$.

$$\Rightarrow \Delta = -\frac14(h^2 + 2r\ell + 2\ell r).$$

To show Z-finiteness, one has

$$\Delta F_f = -e^{-ik\theta}\Big(\underbrace{y^{\frac{k}{2}+2}f_{xx} + y^{\frac{k}{2}+2}f_{yy}}_{=0,\text{ since } f \text{ holomorphic}} + \overbrace{ky^{\frac{k}{2}+1}f_y + \frac{k}{2}\left(\frac{k}{2}-1\right)y^{\frac{k}{2}}f - iky^{\frac{k}{2}+1}f_x}^{if_x = f_y}\Big)$$

$$= \frac{k}{2}\left(\frac{k}{2}-1\right)F_f.$$

To show K-finiteness, one has

$$F_f(g\kappa) = e^{-ik\theta}F_f(g),\quad \text{where } \kappa = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}.$$

· (Moderate growth). By def of "holomorphic at ∞", one has

$$|f(x+iy)| \ll O(1),\quad y\to\infty.$$

exercise!

So $|F_f(g)| = (ad-bc)^{\frac{k}{2}}|ci+d|^{-k}\left|f\left(\frac{ai+b}{ci+d}\right)\right| \ll C_1\|g\|^{C_2k}$.

· (Cusp form). If $f \in S_k(\Gamma)$, then f vanishes at $\infty \Rightarrow f(q) = \underset{=0}{a_0} + a_1q + \cdots$.

By Fourier transform, $0 = a_0 = \int_0^1 f(z+t)\,dt$, $\forall z \in \mathcal{H}$.

However, $F_f\left(\begin{pmatrix}1&t\\&1\end{pmatrix}g\right) = \cancel{\omega(S)e^{-ik\theta}}\, y^{\frac{k}{2}} f(z+t)$. $\square$

Adelic Version & Shimura Varieties.

<u>Recall:</u> Strong Approximation. Let X be an affine scheme over $\mathbb{Z}$. Define:

$$X(\mathbb{A}) := X(\mathbb{R}) \times \prod_p{}' X(\mathbb{Q}_p). \quad \text{with "restrict product topology".}$$

$$\prod_p{}' X(\mathbb{Q}_p) = \{(x_p) \in \prod_p X(\mathbb{Q}_p) : \text{a.e. } x_p \in X(\mathbb{Z}_p)\}.$$

Example: For $X = SL_n$, the special linear group.

- $\mathbb{A}^\times = \mathbb{Q}^\times \cdot (\mathbb{R}_{>0} \times \hat{\mathbb{Z}}^\times)$. and $\mathbb{Q}^\times \hookrightarrow \mathbb{A}^\times$ is discrete.
- $SL_2(\mathbb{A}) = SL_2(\mathbb{Q}) \cdot (SL_2(\mathbb{R}) \times SL_2(\hat{\mathbb{Z}}))$. and $SL_2(\mathbb{Q}) \hookrightarrow SL_2(\mathbb{A})$ is discrete.

Let K_f be a compact subgroup of $SL_2(\mathbb{A}_{fin})$. (<u>Congruence subgroups</u>)

- let Γ_{K_f} be the preimage of $SL_2(\mathbb{R}) \times K_f$ under $SL_2(\mathbb{Q}) \hookrightarrow SL_2(\mathbb{A})$. (e.g. $\Gamma_{SL_2(\hat{\mathbb{Z}})} = SL_2(\mathbb{Z})$).

There is a natural identification:

$$\Gamma_{K_f} \backslash SL_2(\mathbb{R}) \xrightarrow{\sim} SL_2(\mathbb{Q}) \backslash SL_2(\mathbb{A}) / K_f.$$

- let Γ_{K_f} be the preimage of $GL_2(\mathbb{R})^+ \times K_f$ under $GL_2(\mathbb{Q}) \hookrightarrow GL_2(\mathbb{A})$. There are natural identifications:
    - $$\Gamma_{K_f} \backslash GL_2(\mathbb{R})^+ \xrightarrow{1-1} GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / K_f.$$
    - $$\Gamma_{K_f} \backslash \mathcal{H} \xrightarrow{1-1} GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / K_f \times (SO_2(\mathbb{R}) \cdot Z(\mathbb{R})).$$

<u>Proof:</u> Strong approximation & $\mathcal{H} = GL_2(\mathbb{R})^+ / K \cdot Z$. $\square$

<u>Example:</u>
- $SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R}) \simeq SL_2(\mathbb{Q}) \backslash SL_2(\mathbb{A}) / SL_2(\hat{\mathbb{Z}})$.
- Let Γ be a congruence subgroup. Consider $K_f :=$ the profinite group of Γ. i.e. $K_f = \hat{\Gamma} \subseteq GL_2(\hat{\mathbb{Z}})$. Now $\Gamma_{K_f} = GL_2(\mathbb{Q}) \cap (GL_2(\mathbb{R})^+ \times \hat{\Gamma}) = \Gamma$. Hence, The modular curve

$$\Gamma \backslash \mathcal{H} = GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / \hat{\Gamma} \times (K \cdot Z)$$
$$= GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_f) \times \underline{GL_2(\mathbb{R})^+ \times \{\pm 1\}} / \hat{\Gamma} \times \underline{(K \cdot Z)}$$
$$= GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_f) \times \mathcal{H}^{\pm} / \hat{\Gamma}.$$

is a Shimura variety. $\quad$ $\mathcal{H}^{\pm}$: Hermitian symmetric space.

Definition: (Automorphic Forms, Adelic Version)

Goal: $f\in M_k(\Gamma) \rightsquigarrow F_f: GL_2(\mathbb{R})^+\to\mathbb{C} \rightsquigarrow \Phi_f: GL_2(\mathbb{A})\to\mathbb{C}$.

Let $K_f := \hat{\Gamma}$, by strong approximation we have

$$GL_2(\mathbb{A}) = GL_2(\mathbb{Q})\cdot(GL_2(\mathbb{R})^+\times K_f).$$
$$g \longmapsto \gamma\cdot g_\infty\cdot k_f.$$

Define $\Phi_f: GL_2(\mathbb{A})\to\mathbb{C}$, $g\mapsto F_f(g_\infty) = (f|_{[g_\infty]_k})(i)$.

<u>Facts</u>:
- Φ_f is well-defined.
- Properties of Φ_f follows from properties of F_f:
  - $\forall \gamma\in GL_2(\mathbb{Q})$, $\forall g\in GL_2(\mathbb{A})$, $\Phi_f(\gamma g) = \Phi_f(g)$.
  - $\forall k_\infty = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\in SO_2(\mathbb{R})$, $\forall g\in GL_2(\mathbb{A})$, $\Phi_f(gk_\infty) = e^{-ik\theta}\Phi_f(g)$.
  - $\forall \delta = \begin{pmatrix}\lambda & \\ & \lambda\end{pmatrix}\in Z(\mathbb{Q})\backslash Z(\mathbb{A}) \cong \mathbb{Q}^\times\backslash\mathbb{A}^\times$, $\forall g\in GL_2(\mathbb{A})$, $\Phi_f(g\delta) = \omega(\lambda)\Phi_f(g)$.

    Hecke character.
- Under the identification

$$\begin{tikzcd} \Gamma_{K_f}\backslash\mathcal{H} \arrow[r, no head, "1-1"] & GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})/K_f\times(\underline{SO_2(\mathbb{R})\cdot Z(\mathbb{R})}) \\ \mathcal{H} \arrow[u] & GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}) \arrow[u] \end{tikzcd}$$

left action equivarent

invariant

right action invariant

equivarent

$\rightsquigarrow$ <u>Definition:</u> (Automorphic Forms, with out "twists").

An automorphic form on $GL_2(\mathbb{A})$ is a function $\phi: GL_2(\mathbb{A})\to\mathbb{C}$. s.t.

modular forms	$GL_2(\mathbb{R})$ auto. forms	
Level structure	Smoothness & Level structure	· (Smoothness) ◦ (Smooth at inf place): $\forall g\in GL_2(\mathbb{A})$, the induced map $GL_2(\mathbb{R})^+\to\mathbb{C}$, $g_\infty\mapsto\phi(gg_\infty)$ is smooth. ◦ (Locally constant at fin places): $\exists$ compact open subgp $K_f\subseteq GL_2(\mathbb{A}_f)$, s.t. $\forall g\in GL_2(\mathbb{A})$, $\forall k_f\in K_f$, $\phi(gk_f)=\phi(g)$.
		· (Invariant under $GL_2(\mathbb{Q})$). $\forall\gamma\in GL_2(\mathbb{Q})$, $\phi(\gamma g)=\phi(g)$.
twist	twist	· (Central character). $\forall\delta = \begin{pmatrix}\lambda & \\ & \lambda\end{pmatrix}\in Z(\mathbb{Q})\backslash Z(\mathbb{A})$, $\phi(g\delta) = \omega(\lambda)\phi(g)$.

Weights & Holomorphy	Weights & Holomorphy	· (Finiteness) ∘ (K-finite): Let $K = SO_2(\mathbb{R}) \times \prod_p GL_2(\mathbb{Z}_p)$. ϕ is K-finite if $\{k\phi \mid k \in K\}$ spans a finite dim $\mathbb{C}$-linear space. ∘ (Z-finite): Recall $\mathcal{U}(\mathfrak{gl}_2(\mathbb{R})_{\mathbb{C}})$ acts on $C^\infty(GL_2(\mathbb{A}))$ by $(X\cdot\phi)(g) := \frac{d}{dt}\phi(g\exp(tX))\big\vert_{t=0}$ ϕ is Z-finite is ϕ is contained in a finite dim Z-invarant subspace of $C^\infty(GL_2(\mathbb{A}))$.
Growth Condition	Growth Condition	· (Moderate growth). $\exists\, C, N > 0$, s.t. $\lvert\phi(g)\rvert < C \cdot \lVert g \rVert_{\mathbb{A}}^N$, $\forall g \in GL_2(\mathbb{A})$.
Cuspidal	Cuspidal	· (Cusp forms). $\forall g \in GL_2(\mathbb{A})$, $\int_{\mathbb{Q}\backslash\mathbb{A}} \phi\left(\begin{pmatrix}1 & t\\ & 1\end{pmatrix}g\right)dt = 0$.

<u>Theorem</u>: Let $f \in M_k(\Gamma)$, $\omega \in \mathbb{X}$, then $\Phi_f \in \mathcal{A}(GL_2(\mathbb{A}), \omega)$.
If $f \in S_k(\Gamma)$, then $\Phi_f \in \mathcal{A}_0(GL_2(\mathbb{A}), \omega)$.

What Is Automorphic Representations?

There is an action

$$GL_2(\mathbb{A}_f) = \prod_p{}' GL_2(\mathbb{Q}_p) \rightsquigarrow \mathcal{A}(GL_2(\mathbb{A}), \omega)$$

via right translation. But $GL_2(\mathbb{R})$ doesn't quite act on $\mathcal{A}(GL_2(\mathbb{A}), \omega)$.
Instead, it is a so called $(\mathfrak{g} = \mathrm{Lie}(GL_2(\mathbb{R})^+)$ or $\mathrm{Lie}(GL_2(\mathbb{R}))$, $K_\infty = SO_2(\mathbb{R})$ or $O_2(\mathbb{R}))$-module.

$\Rightarrow$ $\mathcal{A}(GL_2(\mathbb{A}), \omega)$ is a "$(\mathfrak{g}, K_\infty) \times GL_2(\mathbb{A}_f)$-module".

$(\mathfrak{g}, K_\infty)$ ↑ Lecture 5 $GL_2(\mathbb{A}_f)$ ↑ Lecture 4, decompose into $GL_2(\mathbb{Q}_p)$.

<u>Definition</u>: An <u>automorphic representation</u> is an irreducible subquotient of $\mathcal{A}(GL_2(\mathbb{A}), \omega)$ as $(\mathfrak{g}, K_\infty) \times GL_2(\mathbb{A}_f)$-module.

Lecture 4.

Representation of $GL_2(\mathbb{Q}_p)$.

<u>Recall</u>: Repn of $G := GL_2(\mathbb{F}_p)$. $(p \neq 2)$.

All the irr. repn. of G can be listed as following:

(Parabolic Inductions).

- 1-dim repn $\mu\circ\det$, where μ is a character on $\mathbb{F}_p^\times$.
- the (twisted) Steinberg repn St_μ, where μ is a character on $\mathbb{F}_p^\times$.
- the principal series $\mathrm{Ind}^G_{\begin{pmatrix} * & * \\ & * \end{pmatrix}}(\chi)$, where χ is a regular character on $\begin{pmatrix} * & \\ & * \end{pmatrix}$, if the maximal subtori in G is isomorphic to $(\mathbb{F}_p^2)^\times$.

(Weil Repns $\rightsquigarrow$ Cuspidal Repns).

- the theta series π_χ, where χ is a regular character on $\left\{\begin{pmatrix} a & \delta\beta \\ \beta & a \end{pmatrix} \in G\right\}$, $\delta \in \mathbb{F}_p^\times \setminus (\mathbb{F}_p^\times)^2$, if the maximal subtori in G is isomorphic to $(\mathbb{F}_p[\sqrt{\delta}])^\times$.

  Recall: A character χ is called regular if $\chi \neq \chi^\tau$, where τ is the Galois involution.

Our Goal:

Irr. (admissable) repn of $GL_2(\mathbb{Q}_p)$
$$\boxed{\begin{cases} \text{1-dim repns: } \mu\circ\det,\ \text{where } \mu: \mathbb{Q}_p^\times \to \mathbb{C}^\times. \\ \text{principal series: } B(\chi_1,\chi_2) := \mathrm{Ind}_P^G(\chi_0),\ \text{where } \chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}. \\ \text{Steinberg repns: } 0 \to \mu\circ\det \to \mathrm{Ind}_P^G(\chi_0) \to St_\mu \to 0,\ \text{where } \chi_1\chi_2^{-1} = |\cdot|^{\pm 1}. \\ \text{supercuspidal repns: others (not subquotients of } B(\chi_1,\chi_2)). \end{cases}}$$

Definition: (Smooth & Admissable).

Let $\pi: GL_2(\mathbb{Q}_p) \to V$ (or (π, V) simply) be a repn of $GL_2(\mathbb{Q}_p)$ over $\mathbb{C}$-linear space.

- π is smooth if: $\forall v \in V$, $\{g \in GL_2(\mathbb{Q}_p) : \pi(g)v = v\}$ is compact open.
- π is admissable if: π is smooth and $\forall$ compact open subgp $K \subseteq GL_2(\mathbb{Q}_p)$, the space V^K is finite dim. (It is enough to check this if K run out all $K_n := \begin{pmatrix} 1+p^n\mathbb{Z}_p & p^n\mathbb{Z}_p \\ p^n\mathbb{Z}_p & 1+p^n\mathbb{Z}_p \end{pmatrix}$, this is a system of open (compact) neighborhood of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{Z}_p)$).

Fact: Irr. + smooth $\Rightarrow$ admissable, (Non-trivial!) and has a "central character" $\omega: \mathbb{Q}_p^\times \to \mathbb{C}^\times$, s.t. $\pi\begin{pmatrix} a & \\ & a \end{pmatrix} v = \omega(a) v$. (By Schur lemma below, since $\begin{pmatrix} a & \\ & a \end{pmatrix} \in Z$).

Proposition: (Some important properties).

- (Schur Lemma). Let (π, V) smooth irr. of $GL_2(\mathbb{Q}_p)$, then $\dim_{\mathbb{C}}(\mathrm{End}_{GL_2(\mathbb{Q}_p)}(V)) = 1$.
- (π, V) smooth. Then $V = \bigoplus_\rho \underbrace{V(\rho)}_{\text{the } \rho\text{-isotypic component}}$, where

  $\rho \in \{$finite dim irr repn of K whose kernel is open$\}$.

  So (π, V) is admissable iff $\forall \rho$. $\dim_{\mathbb{C}}(V(\rho)) < \infty$. (so admissable $\not\Rightarrow$ irr.)

Proof: Step 1. If (π, V) smooth irr, then $\dim_{\mathbb{C}}(V)$ is countable.

Pf: Use the Iwasawa decomposition

$$GL_2(\mathbb{Q}_p) = \{ \underbrace{\begin{pmatrix} p^{e_1} & \\ & p^{e_2} \end{pmatrix}\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}}_{P = \begin{pmatrix} * & * \\ & * \end{pmatrix}} \underbrace{k}_{K = GL_2(\mathbb{Z}_p)} : e_1 > e_2 \in \mathbb{Z},\ k \in GL_2(\mathbb{Z}_p),\ u = \sum_{-N}^{e_1-e_2-1} u_i p^i \}.$$

Now, π irr $\Rightarrow \forall\, 0 \neq v \in V$, $\mathrm{span}\{\pi(g)v : g = pk \in GL_2(\mathbb{Q}_p)\} = V$.

By smoothness, $K' := \{g \in GL_2(\mathbb{Q}_p) : \pi(g)v = v\} \cap GL_2(\mathbb{Z}_p) = \{g \in GL_2(\mathbb{Z}_p) : \pi(g)v = v\}$ is compact. $\Rightarrow [K : K'] < \infty$.

So $\pi(K)v$ is a finite set. Hence

$$V = \mathrm{span}\{\pi(p)v\} \times \{\text{Finite Set}\}$$

is spanned by a countable set.

Step 2. Prove Schur lemma.

Pf: If there exists $L : V \to V$ s.t. $\forall z \in \mathbb{C}$, $L - z\mathrm{Id} \neq 0$. Then:

π irr $\Rightarrow L - z\mathrm{Id}$ must be isomorphisms. Consider the uncountable set

$$\{(L - z\mathrm{Id})^{-1} v : z \in \mathbb{C}\} \subseteq V := \mathrm{span}_{i \in \mathbb{Z}}\{e_i\}$$

$\Rightarrow \exists$ linear relation $0 = \sum_{i=1}^{N} a_i (L - z_i \mathrm{Id})^{-1} v \Rightarrow \prod_{i=1}^{N} (L - w_i \mathrm{Id}) v = 0$.

$\rightsquigarrow L - w_i\mathrm{Id}$ is not injective for some w_j. A contradiction! □

CLASSIFICATIONS:

- Principal Series. (Parabolic Induction)

Definition: Let $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ be quasi characters. Let

Normalize, to make the induced repn of unitary repns also unitary.

$$B(\chi_1, \chi_2) := \{ f : GL_2(\mathbb{Q}_p) \to \mathbb{C} \text{ locally constant} \mid f(\begin{pmatrix} a & b \\ & c \end{pmatrix} g) = \chi_1(a)\chi_2(c) \left|\frac{a}{c}\right|^{\frac{1}{2}} f(g) \}$$

$$\left(= \mathrm{Ind}_P^{GL_2(\mathbb{Q}_p)}\left(\begin{pmatrix} a & b \\ & c \end{pmatrix} \mapsto \chi_1(a) \cdot \chi_2(c) \left|\frac{a}{c}\right|^{\frac{1}{2}}\right)\right)$$

Theorem: There is a left action

$\forall g \in GL_2(\mathbb{Q}_p)$, $(g \cdot f)(x) := f(xg)$.

It makes $B(\chi_1, \chi_2)$ an admissable repn.

Proof: $GL_2(\mathbb{Q}_p) = PK \Rightarrow$ we only need to study $f(k)$, $k \in K = GL_2(\mathbb{Z}_p)$.

Step 1. Smoothness: ($\forall f$, $U_f := \{g \mid f(xg) = f(x)\}$ compact open).

Pf: $f \in B(\chi_1, \chi_2) \Rightarrow \forall p \in P, \forall g \in GL_2(\mathbb{Q}_p)$, $f(pg) = \chi_0(p) f(g)$ for some χ_0.

f locally constant $\Rightarrow \forall k$, $f(kK_{n_k}) = f(k)$ for some n_k.

$\Rightarrow U_f = \bigcap_k K_{n_k} \subseteq K_{n_k} \subseteq K \subseteq \bigcup_k k K_{n_k}$

$\Rightarrow U_f = \bigcap_{k\in A} K_{n_k} \neq \phi$ — finite union, since K is compact. So k just needs to run out a finite set A.

since: · K_n is a compact neighborhood system $\Rightarrow U_f$ compact.

· can be a finite intersection $\Rightarrow U_f$ open.

Step 2. Admissable. ($\forall K' \subseteq K$ compact, $B(\chi_1,\chi_2)^{K'}$ has finite dim).

Pf: $\forall f \in B(\chi_1,\chi_2)^{K'}$, $f(kk') = f(k)$, where $k\in K$, $k'\in K'$.

$[K:K'] < \infty \Rightarrow$ values of $f(k)$ depends only on what coset of K' it is in.

$\Rightarrow f(k)$ is determined by finite things. $\square$

Case I: $\chi_1\chi_2^{-1} = |\cdot|^{\pm 1}$. $\longrightarrow$ Steinberg Repns, which are ∞-dim.

Proposition: If $\chi_1\chi_2^{-1} = |\cdot|^{-1}$, then $B(\chi_1,\chi_2)$ has a 1-dim $GL_2(\mathbb{Q}_p)$-subspace and the quotient is irr., If $\chi_1\chi_2^{-1} = |\cdot|$, then $B(\chi_1,\chi_2)$ has an irr. subrepn with codim 1.

The remaining spaces above are called Steinberg repns.

Proof: Only prove the first one. Let $\chi = \chi_1|\cdot|^{\frac{1}{2}} = \chi_2|\cdot|^{-\frac{1}{2}}$.

Now $\mathbb{C}[\chi\circ\det]$ is a dim 1 inv. subspace of $B(\chi_1,\chi_2)$, since $\forall g\in GL_2(\mathbb{Q}_p)$,

$$\chi(\det(\begin{pmatrix} a & x \\ & b\end{pmatrix} g)) = \chi(ab)\chi(\det(g)) = \chi_1(a)\chi_2(b)\left|\frac{a}{b}\right|^{\frac{1}{2}}(\chi\circ\det)(g).$$

and $\forall g$, $g(\chi\circ\det) = \underbrace{[\chi\circ\det(g)]}_{\in\mathbb{C}^\times}\cdot(\chi\circ\det)$.

To prove the quotient is irr, one needs Whittaker functionals. $\square$

There are no any other principal series repns which are reducible, besides for the ones mentioned above. For a proof, See Bump p. 475.

Case II: $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$.

Fact: $B(\chi_1,\chi_2)$ is irr.

Theorem: (Classification of principal series) $\forall \chi_1,\chi_2: \mathbb{Q}_p^\times \to \mathbb{C}^\times$,

- If $B(\chi_1,\chi_2)$ is irr, then $B(\chi_1,\chi_2) \cong B(\chi_2,\chi_1)$. $\rightsquigarrow$ the induced Steinberg repns are still isomorphic (if exist).
- If $\chi = \chi_1 = \chi_2$, then $B(\chi,\chi) = \chi\otimes B(1,1)$ is irr.
- If $\chi_1 \neq \chi_2$, $\mathrm{Hom}_{rep}(B(\chi_1,\chi_2), B(\eta_1,\eta_2)) = \begin{cases} \mathbb{C}, & (\chi_1,\chi_2) = (\eta_1,\eta_2) \text{ or } (\eta_2,\eta_1). \\ 0, & \text{otherwise}.\end{cases}$

- Supercuspidal Repns. (Difficult to classify)

Definition: (Jacquet modules). Define a Functor

$J: \text{Repns} \to \begin{pmatrix} * & \\ & * \end{pmatrix}\text{-Mod}, \quad (\pi, V) \mapsto J(V) := V/\text{span}\{\pi\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}v - v \mid v \in V, x \in \mathbb{Q}_p\}.$

Remark: • Well-definedness: $\forall \begin{pmatrix} a & \\ & b \end{pmatrix}, \pi\begin{pmatrix} a & \\ & b \end{pmatrix}\left(\pi\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}v - v\right)$, $a, b \in \mathbb{Q}_p^\times$

$$= \pi\begin{pmatrix} 1 & \frac{a}{b}x \\ & 1 \end{pmatrix}\pi\begin{pmatrix} a & \\ & b \end{pmatrix}v - \pi\begin{pmatrix} a & \\ & b \end{pmatrix}v$$

• $w \in \text{Span}\{\pi\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}v - v\} \Leftrightarrow \int_U \pi(u) w \, du = 0$ for some compact open subgp $U \subseteq \mathbb{Q}_p$.

• J is exact.

Theorem: Let (π, V) be a irr. admissable repn of $GL_2(\mathbb{Q}_p)$. Then (π, V) is supercuspidal iff $J(V) = 0$.

Ramification Theory.

Definition: (π, V) is unramified if $V^K \neq 0$, a non-zero element in V^K is called a spherical vector. Otherwise is ramified.

Example: $|\cdot| \circ \det = \text{id}$ as scalar multiplication is unramified.

Remark: • (π, V) irr. admissable, then $\dim_{\mathbb{C}}(V^K) \leq 1$. (Not easy!)

• The only possible unramified repns are in the principal series $B(\chi_1, \chi_2)$ where χ_1, χ_2 are unramified. In that case, we have a spherical vector

$$f\left(\begin{pmatrix} a & x \\ & b \end{pmatrix} k\right) = \chi_1(a)\chi_2(b)\left|\frac{a}{b}\right|^{\frac{1}{2}}.$$

So there is a classification:

∘ All spherical repns of $GL_2(\mathbb{Q}_p)$ are following:

1). The inf-dim ones are $B(\chi_1, \chi_2)$, s.t. $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$.

2). The fin-dim ones are 1-dim, They are $\chi \circ \det$ for unramified χ.

Lecture 5

Representation of $(\mathfrak{g}, K)$-module.

Set Up: Throughout this part, $(G, K) := (GL_2(\mathbb{R})^+, SO_2(\mathbb{R}))$ or $(GL_2(\mathbb{R}), O_2(\mathbb{R}))$. Let $\mathfrak{g} := \text{Lie}(G)$, $\mathfrak{k} := \text{Lie}(K)$. In these cases, $\mathfrak{g}_{\mathbb{C}} = \text{Mat}_2(\mathbb{C})$.

Definition: A $(\mathfrak{g}, K)$-module is a $\mathbb{C}$-linear space V together with Lie group repn π_K of K and Lie algebra repn $\pi_{\mathfrak{g}}$ of $\mathfrak{g}$ s.t.:

1). V decomposes into a countable direct sum $V=\oplus_i V_i$ of finite dim irr. K-subrepns.

2). For $X\in\mathfrak{k}$, $v\in V$, we have $\pi_{\mathfrak{g}}(X)v = Xv := \frac{d}{dt}(\pi_K(\exp(tX))v)\big|_{t=0}$.

3). For $k\in K$, $X\in\mathfrak{g}$, $v\in V$, we have $\pi_K(k)\pi_{\mathfrak{g}}(X)\pi_K(k^{-1})v = \pi_{\mathfrak{g}}(Ad(k)X)v$.

Remark: 1) + 2) $\Rightarrow$ 3) when $(G,K)=(GL_2(\mathbb{R})^+, SO_2(\mathbb{R}))$. But not right in the other.

Definition: A $(\mathfrak{g},K)$-module is admissable if in the decomposition $V=\oplus_i V_i$, no isomorphism type of irr. K-repn occurs with infinite multiplicity.

Example: $(G,K)=(GL_2(\mathbb{R})^+, SO_2(\mathbb{R}))$, f a cusp form. $\rightsquigarrow F_f\in\mathcal{A}_0(\Gamma\backslash G,\omega)$.

In this example, $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})F_f$ is an irr. admissable $(\mathfrak{g},K)$-module.

Remark: $\mathcal{A}_0$ VS L_0^2.

$$(\mathfrak{g},K)\circlearrowright \mathcal{A}_0(\Gamma\backslash G,\omega) \hookrightarrow L_0^2(\Gamma\backslash G,\omega)\circlearrowleft G$$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \|$ spectral theory

$\forall i$, Let $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \hat{\oplus}_i H_i$

H_i^{fin} := K-finite vectors in H_i, $\quad\quad$ Hilbert direct sum of irr. "complete" G-subrepns.

Then: ∘ $H_i^{fin}\subseteq H_i$ dense.

∘ $H_i^{fin}\subseteq\mathcal{A}_0(\Gamma\backslash G,\omega)$. Hence H_i^{fin} is an irr. admissable $(\mathfrak{g},K)$-module. (that is, H_i is an "irr. admissable G-repn").

So: $\quad \mathcal{A}_0(\Gamma\backslash G,\omega)=\oplus_i H_i^{fin}$, an algebraic direct sum.

To sum up, $\left\{\begin{matrix}\text{irr. admissable}\\ (\mathfrak{g},K)\text{-mod in } \mathcal{A}_0\end{matrix}\right\} \xleftrightarrow{1-1} \left\{\begin{matrix}\text{irr. admissable}\\ G\text{-repn in } L_0^2\end{matrix}\right\}$, $H_i^{fin}\longleftrightarrow H_i$.

$\quad\quad\quad\quad\quad\quad$ algebraically, easy! $\quad\quad\quad$ analytically, unfamiliar.

CLASSIFICATIONS:

Let V be an irr. admissable $(\mathfrak{g},K)$-module. Recall that irr. $K(\cong\mathbb{R}/\mathbb{Z})$-repns are 1-dim given by

$$\sigma_k:\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\longmapsto e^{-ik\theta},\quad k\in\mathbb{Z}.$$

Use Peter-Weyl we get: $V=\bigoplus_{k\in\mathbb{Z}=\hat{K}} V(k)$ $\quad$ σ_k-isotypic part. since admissable, $V(k)$ are finite dim. in fact, $\dim_{\mathbb{C}}(V(k))\le 1$.

Recall: $\mathfrak{z}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $\mathfrak{r}=\frac{1}{2}\begin{pmatrix}1&i\\i&-1\end{pmatrix}$, $\mathfrak{l}=\frac{1}{2}\begin{pmatrix}1&-i\\-i&-1\end{pmatrix}$, $\mathfrak{h}=\begin{pmatrix}&-i\\i&\end{pmatrix}\in\mathfrak{g}_{\mathbb{C}}$.

$\mathcal{U}(\mathfrak{g}_{\mathbb{C}})\cong\mathbb{C}[\mathfrak{z},\mathfrak{r},\mathfrak{l},\mathfrak{h}]/\langle \mathfrak{z}\mathfrak{r}-\mathfrak{r}\mathfrak{z}, \mathfrak{r}\mathfrak{l}-\mathfrak{l}\mathfrak{r}-\mathfrak{h},\cdots\rangle$

Then: $Z(\mathcal{U}(\mathfrak{g}_{\mathbb{C}}))=\mathbb{C}[\mathfrak{z},\Delta]$, where $\Delta=-\frac{1}{4}(\mathfrak{h}^2+2\mathfrak{r}\mathfrak{l}+2\mathfrak{l}\mathfrak{r})$.

<u>Proposition:</u> V irr. admissable $(\mathfrak{g}, K)$-module. Then:

- $\forall x \in V(k)$, $hx = kx$. Hence $V(k) = \{x \in V \mid hx = kx\}$.
- If $x \in V(k)$, then $rx \in V(k+2)$, $lx \in V(k-2)$.
- If $0 \neq x \in V(k)$, then $V(k) = \mathbb{C}x$; $V(k+2n) = \mathbb{C} r^n x$; $V(k-2n) = \mathbb{C} l^n x$; $V = \bigoplus_{n \in \mathbb{Z}} V(k+2n)$.

Sum up, V looks like:

$$\begin{tikzcd}[column sep=small] \cdots \oplus & V(k-4) \arrow[rr, bend left, "r"] & \oplus & V(k-2) \arrow[ll, bend left, "l"] \arrow[rr, bend left, "r"] & \oplus & V(k) \arrow[ll, bend left, "l"] \arrow[rr, bend left, "r"] & \oplus & V(k+2) \arrow[ll, bend left, "l"] \arrow[rr, bend left, "r"] & \oplus & V(k+4) \arrow[ll, bend left, "l"] & \oplus \cdots \end{tikzcd}$$

dim $= 1$.

- The Casimir element Δ acts on V by the scalar ω. For $x \in V(k)$,

$$lrx = \left(-\omega - \tfrac{k}{2}\left(1 + \tfrac{k}{2}\right)\right)x \quad ; \quad rlx = \left(-\omega + \tfrac{k}{2}\left(1 - \tfrac{k}{2}\right)\right)x.$$

Moreover. $\begin{cases} \text{If } 0 \neq x \in V(k) \text{ but } rx = 0, \text{ then } \omega = -\frac{k}{2}\left(1 + \frac{k}{2}\right); \\ \text{If } 0 \neq x \in V(k) \text{ but } lx = 0, \text{ then } \omega = \frac{k}{2}\left(1 - \frac{k}{2}\right). \end{cases}$

<u>Proof:</u> • $\forall x \in V(k) \subseteq V$,

$$(ih)x = \frac{d}{dt}\left(\pi_K\left(\exp\left(t\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\right)x\right)\Big|_{t=0} = \lim_{\Delta t \to 0} \frac{\pi_K\left(1 + \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} -t^2 & 0 \\ 0 & -t^2 \end{pmatrix} + \cdots + \frac{1}{n!}\begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}^n + \cdots\right)x - x}{t}$$

$$= \lim_{\Delta t \to 0} \frac{\sigma_k\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}x - x}{t} = \lim_{\Delta t \to 0} \frac{(e^{ikt} - 1)x}{t} = ikx. \quad \Rightarrow hx = kx.$$

- Recall that $[h, r] = 2r$, we have $hrx = rhx + [h, r]x = (k+2)rx$.
  $\Rightarrow rx \in V(k+2)$. Similarly, $lx \in V(k-2)$.
- Use facts above, we have $\mathbb{C}x \subseteq V(k)$, $\mathbb{C}r^n x \subseteq V(k+2n)$, $\mathbb{C}l^n x \subseteq V(k-2n)$. This implies the claim since V is irr.
- Recall that $[r, l] = 2h$, we have $-4\Delta = h^2 + 2h + 4lr$. So:

$$lrx = -\omega x - \tfrac{1}{4}k^2 x - \tfrac{1}{2}kx = \left(-\omega - \tfrac{k}{2}\left(1 + \tfrac{k}{2}\right)\right)x. \qquad \square$$

<u>Corollary:</u> The isomorphism class of irr. admissable $(\mathfrak{g}, K)$-modules is determined by: 1) the eigenvalue of z; 2) the eigenvalue ω of Δ; 3) the K-type $\Sigma(V) := \{n \in \mathbb{Z} \mid V(n) \neq 0\}$. (<u>it contains even (odd)</u>).

If write $\omega = \frac{k}{2}\left(1 - \frac{k}{2}\right)$ with $k \in \mathbb{C}$, write $\varepsilon := \begin{cases} 0, & \text{even} \\ 1, & \text{odd} \end{cases}$. We have:

- If $k \notin \mathbb{Z}$, or $k \equiv \varepsilon \pmod 2$, then $\Sigma(V) = \{n \in \mathbb{Z} \mid n \equiv \varepsilon \pmod 2\}$.
- If $k \in \mathbb{Z}$ and $k \not\equiv \varepsilon \pmod 2$, then the K-type can be either

- $\Sigma_+(k) = \{ n \geq |k| : n \equiv k \pmod 2 \}$.
- $\Sigma_0(k) = \{ -|k| < n < |k| : n \equiv k \pmod 2 \}$.
- $\Sigma_-(k) = \{ n \leq -|k| : n \equiv k \pmod 2 \}$.

The case $s=0$ is exceptional because in that case $\Sigma_0 = \phi$.

<u>Questions</u>: · Do the $(\mathfrak{g}, K)$-modules described above actually exist?

· Can each such $(\mathfrak{g}, K)$-module be realized in the K-finite vectors in some irr. admissable repn (π, V) of G?

Let ε be above. Let $s_1, s_2 \in \mathbb{C}$. Let $\underset{\substack{\uparrow \\ \text{eigenvalue of } \Delta}}{\omega} := s(1-s)$, $\underset{\substack{\uparrow \\ \text{eigenvalue of } \mathfrak{z}}}{\eta} := s_1 + s_2$, where $s = \frac{s_1 - s_2 + 1}{2}$. (on $\pi(s_1, s_2, \varepsilon)$).

Now construct an admissable repn $\pi(s_1, s_2, \varepsilon)$ of G on a Hilbert space:

Given a character $\chi : \underset{\substack{\| \\ \begin{pmatrix} * & * \\ & * \end{pmatrix} \subseteq GL_2(\mathbb{R})^+}}{P} \to \mathbb{C}$, $\begin{pmatrix} y_1 & x \\ & y_2 \end{pmatrix} \mapsto (\operatorname{sgn}(y_1))^{\varepsilon} \cdot |y_1|^{s_1} \cdot |y_2|^{s_2}$.

Let

$$\underset{\substack{\| \\ \operatorname{Ind}_P^G(\chi)}}{H(s_1, s_2, \varepsilon)} := \left\{ \begin{array}{c} \text{smooth function} \\ f : G \to \mathbb{C} \end{array} \;\middle|\; \begin{array}{l} \cdot\ f\left(\begin{pmatrix} y_1 & x \\ & y_2 \end{pmatrix} g \right) = y_1^{s_1 + \frac{1}{2}} y_2^{s_2 - \frac{1}{2}} f(g),\ \forall y_1, y_2 > 0,\ \forall g \in G \\ \cdot\ f\left(\begin{pmatrix} -1 & \\ & 1 \end{pmatrix} g \right) = (-1)^{\varepsilon} f(g),\ \forall g \in G \end{array} \right\}.$$

<u>Facts</u>: Let $H(s_1, s_2, \varepsilon)$ as above.

- The G action $gf(x) = f(xg)$ makes $H(s_1, s_2, \varepsilon)$ a G-repn.
- Since <u>$\forall f \in H(s_1, s_2, \varepsilon)$ is determined by its values on K</u> (by Iwasawa), we can give $H(s_1, s_2, \varepsilon)$ a Hermitian inner product
$$\langle f_1, f_2 \rangle := \frac{1}{2\pi} \int_0^{2\pi} f_1(\kappa_\theta) \overline{f_2(\kappa_\theta)}\, d\theta.$$
The completion of $H(s_1, s_2, \varepsilon)$ is a Hilbert space and denoted by $\tilde{H}(s_1, s_2, \varepsilon)$.
Elements in $\underset{\circlearrowleft G}{H(s_1, s_2, \varepsilon)} \subseteq \underset{\circlearrowleft G}{\tilde{H}(s_1, s_2, \varepsilon)}$ play the role of "smooth vectors" for the G-repn.

Define: $\pi(s_1, s_2, \varepsilon) := \tilde{H}(s_1, s_2, \varepsilon)^{\text{fin}}$ (space of K-finite vectors). This is a $(\mathfrak{g}, K)$-module:

<u>Theorem</u>: For $s = \frac{s_1 - s_2 + 1}{2}$, $\omega = s(1-s)$, $\eta = s_1 + s_2$,

1) $\Delta, \mathfrak{z}$ acts via scalars on $\pi(s_1, s_2, \varepsilon)$ with eigenvalues ω, η.
2) Suppose s is not of the form $\frac{k}{2}$, $k \equiv \varepsilon \pmod 2$, then $\pi(s_1, s_2, \varepsilon)$ is irr.
The set of K-types is $\{ n \in \mathbb{Z} \mid n \equiv \varepsilon \pmod 2 \}$.

3) Suppose $s=\frac{k}{2}$, $k\equiv\varepsilon \pmod 2$, $k\geq 1$, then $\pi(s_1,s_2,\varepsilon)$ has two irr. invariant space π_+ & π_-. The set of K-types of $\pi_\pm$ is $\Sigma_\pm(k)$.

Moreover, the quotient $\pi/(\pi_+\oplus\pi_-)$ is irr, unless $k=1$, in which case it is 0, the set of K-types is $\Sigma_0(k)$.

4) Suppose $s=1-\frac{k}{2}$, $k\equiv\varepsilon \pmod 2$, $k\geq 1$, then $\pi(s_1,s_2,\varepsilon)$ has an invariant subspace π' whose set of K-types is $\Sigma_0(k)$. Here π' is irr, unless $k=1$, in which case it is 0.

Moreover, the quotient π/π' decomposes into two irr. invariant subspaces π'_+ & π'_-, the set of K-types of $\pi'_\pm$ is $\Sigma_\pm(k)$.

<u>Remark</u>:
- $\pi(s_1,s_2,\varepsilon)$ in (2) are called <u>principal series</u>.
- $\pi/(\pi_+\oplus\pi_-)$ in (3), π' in (4) are finite dim.
- π_+, π_- in (3), π'_+, π'_- in (4) are called <u>discrete series</u>.
- These (1, 2, 3, 4) are all of the irr. admissable $(\mathfrak{g},K)$-modules.

Classification of $(\mathfrak{g},K)$-modules for $GL_2(\mathbb{R})^+$ imply the classification of unitary repns of $GL_2(\mathbb{R})^+$.

"topological" repn of $GL_2(\mathbb{R}) \rightsquigarrow$ "algebraic" $(\mathfrak{g},K)$-module.

<u>Proof</u>: By K-finiteness, $\exists\, f_t\in\widetilde{H}(s_1,s_2,\varepsilon)$ s.t. $f_t(\kappa_\theta)=e^{it\theta}$. Since

$$f_t(\kappa_{\theta+\pi})=(-1)^\varepsilon f_t(\kappa_\theta),$$

we must have $t\equiv\varepsilon \pmod 2$. Now by definition

$$f_t\left(\begin{pmatrix}u&\\&u\end{pmatrix}\begin{pmatrix}y^{\frac12}&xy^{-\frac12}\\&y^{-\frac12}\end{pmatrix}\kappa_\theta\right)=u^{s_1+s_2}y^s e^{it\theta}.$$

<u>Step 1</u>: $h f_t=t f_t$; $\mathcal{L}f_t=(s+\frac{t}{2})f_{t+2}$; $\mathcal{R}f_t=(s-\frac{t}{2})f_{t-2}$. and $\Delta f_t=\omega f_t$, $\mathfrak{z}f_t=\eta f_t$.

<u>Pf</u>: For example,

$$\mathcal{L}f_t=e^{2i\theta}\left(-iy\frac{\partial f_t}{\partial x}+y\frac{\partial f_t}{\partial y}+\frac{1}{2i}\frac{\partial f_t}{\partial\theta}\right)$$
$$=e^{2i\theta}\left(s\cdot \underline{u^{s_1+s_2}y^s e^{it\theta}}+\frac12 t\underline{u^{s_1+s_2}y^s e^{it\theta}}\right)$$
$$=e^{2i\theta}(s+\tfrac{t}{2})f_t=(s+\tfrac{t}{2})f_{t+2}.$$

Moreover, $\mathfrak{z}=u\frac{\partial}{\partial u}$; $-4\Delta f_t=\left(t^2+2(s+\frac{t}{2}-1)(s-\frac{t}{2})+2(s-\frac{t}{2}-1)(s+\frac{t}{2})\right)f_t$.

<u>Step 2</u>: proof of (3). (omit (2) & (4)). By (1) we have $\mathcal{R}f_k=\mathcal{L}f_{-k}=0$.

Consequently, $\bigoplus_{n=0}^{\infty}\mathcal{L}^n f_k$ & $\bigoplus_{n=0}^{\infty}\mathcal{R}^n f_{-k}$ are inv. submodules whose sets of K-types are $\Sigma_+(k)$ & $\Sigma_-(k)$. By corollary above, π_+, π_- and the quotient are all irr. $\square$

Lecture 6

Global Automorphic Repn Theory

Recall: $\mathcal{A}(GL_2(\mathbb{A}), \omega) \circlearrowleft (\mathfrak{g}, K_\infty) \times GL_2(\mathbb{A}_f)$ - module.

<u>Definition</u>: (Admissable repns) Let $K := K_\infty \times K_f = SO_2(\mathbb{R}) \times \prod_p GL_2(\mathbb{Z}_p) \subseteq GL_2(\mathbb{A})$ be the maximal compact subgroup. A $(\mathfrak{g}, K_\infty) \times GL_2(\mathbb{A}_f)$-module (π, V) is <u>admissable</u> if:

- every vector is K-finite.
- $V = \bigoplus_i V_i$ decomposes into a countable direct sum of finite dim irr. K-subrepns and no irr. repn of K occurs with infinite multiplicity.

<u>Fact</u>: Automorphic repns are irr. admissable repns.

<u>Definition</u>: The global Hecke algebra

$$\mathcal{H} := \mathcal{H}_\infty \otimes \left(\bigotimes_p{}' \mathcal{H}_p \right).$$

where:
- $\mathcal{H}_p = \mathcal{H}_{GL_2(\mathbb{Q}_p)} := C_c^\infty(GL_2(\mathbb{Q}_p))$.
- $\mathcal{H}_\infty :=$ the algebra of compactly supported distributions on $GL_2(\mathbb{R})$.
- $\otimes'$: the restrict tensor. for every finite set S of finite places, let $\mathcal{H}_S := (\bigotimes_{p \in S} \mathcal{H}_p) \otimes (\bigotimes_{p \notin S} \{\mathbb{1}_{GL_2(\mathbb{Z}_p)}\})$. then $\bigotimes_p' \mathcal{H}_p := \varinjlim_S \mathcal{H}_S$.
  Indeed, $\bigotimes_p' \mathcal{H}_p = C_c^\infty(GL_2(\mathbb{A}_f))$.

So, the concept of admissable $(\mathfrak{g}, K_\infty) \times GL_2(\mathbb{A}_f)$-module induces the concept of "admissable" $\mathcal{H}$-module.

<u>Theorem</u>: (Tensor Product Theorem, Flath).

Let (π, V) be an irr. admissable $(\mathfrak{g}, K_\infty) \times GL_2(\mathbb{A}_f)$-module. Then $\exists$
- an irr. admissable $(\mathfrak{g}, K_\infty)$-module (π_∞, V_∞),
- $\forall p$, an irr. admissable repn (π_p, V_p) of $GL_2(\mathbb{Q}_p)$. s.t. V_p contains a non-zero $GL_2(\mathbb{Z}_p)$-fixed vector ξ_p (i.e. unramify and has a spherical vector), for almost all p.

- $\pi \simeq \pi_\infty \otimes \left({\bigotimes_p}' \pi_p\right)$.

<u>Remark</u>: Let S be finite set of finite places containing all p where π_p is non-spherical (ramify), and let $\pi_S := \left(\bigotimes_{p\in S} \pi_p\right) \otimes \left(\bigotimes_{p\notin S} \{\xi_p\}\right)$. Then we define

$$ {\bigotimes_p}' \pi_p := \varinjlim_S \pi_S . $$

<u>Corollary</u>: (Multiplicity One).

Let (π, V), (π', V') be irr. admissable $(\mathfrak{g}, K_\infty) \times GL_2(\mathbb{A}_f)$-modules appearing $\mathcal{A}_0(GL_2(\mathbb{A}), \omega)$. Assume $\pi_p \simeq \pi_p'$ for almost all p, then $\pi \simeq \pi'$.

<u>From Modular Forms</u>: Let $f \in S_k(N) \rightsquigarrow \Phi_f \in \mathcal{A}_0(GL_2(\mathbb{A}), \omega)$.

<u>Theorem</u>:
- Recall Hecke operators $T_p := \mathbb{1}_{GL_2(\mathbb{Z}_p)\left(\begin{smallmatrix} p & \\ & 1\end{smallmatrix}\right)GL_2(\mathbb{Z}_p)}$; $R_p := \mathbb{1}_{GL_2(\mathbb{Z}_p)\left(\begin{smallmatrix} p & \\ & p\end{smallmatrix}\right)GL_2(\mathbb{Z}_p)}$.
  Then $T_p' := T_p \otimes \left(\bigotimes_{\ell\neq p} \mathbb{1}_{GL_2(\mathbb{Z}_\ell)}\right)$, $R_p' := R_p \otimes \left(\bigotimes_{\ell \neq p} \mathbb{1}_{GL_2(\mathbb{Z}_\ell)}\right)$ are elements in $\mathcal{H}$.
  Now: $T_p'(\Phi_f) = p^{1-\frac{k}{2}} \Phi_{T_p f}$; $R_p'(\Phi_f) = \Phi_f$.
  Hence, if f is an eigenform for all T_p ($p \nmid N$), Then Φ_f lies in a unique automorphic repn $\pi_f \subseteq \mathcal{A}_0(GL_2(\mathbb{A}), \omega)$. (By Flath Thm).
- Let $\pi_f = \pi_{f,\infty} \otimes \left({\bigotimes_p}' \pi_{f,p}\right)$ above. Then:
  - $\pi_{f,\infty}$ is a discrete series.
  - For $p \nmid N$, $\pi_{f,p}$ is a spherical principal series.
  - If $p \mid N$, complicated! But there is an algorithm to compute $\pi_{f,p}$.

GL_2-LANGLANDS:

<u>Local Langlands</u>: (Harris-Taylor, Henniart, Scholze) Roughly,

$$ \{\text{irr. admissable repn of } GL_2(\mathbb{Q}_p)\} \xleftrightarrow{LL_2} \{\text{"2-dim repn of } Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\text{"}\}. $$

Indeed, "Weil-Deligne repn". No topology of base field is involved in these repns!

Consider the exact sequence:

$$ \begin{array}{ccccccccc} 0 \to & I_{\mathbb{Q}_p} & \to & Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) & \xrightarrow{\alpha} & Gal(\mathbb{Q}_p^{ur}/\mathbb{Q}_p) & \to 0 \\ & \| & & & & \wr\| & \\ & Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{ur}) & & & & Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p) & \\ & & & & & \wr\| & \\ & & & & & \hat{\mathbb{Z}} \cdot \mathrm{frob}_p . & \end{array} $$

The <u>Weil group</u> $W_{\mathbb{Q}_p} \subseteq Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is $\alpha^{-1}(\mathbb{Z})$. A <u>Weil-Deligne repn</u> is a triple (V, r, N), where:

- V is finite dim $\mathbb{C}$-linear space.

- $r: W_{\mathbb{Q}_p} \to GL(V)$ is a repn s.t.
  - $r|_{I_{\mathbb{Q}_p}}$ has open kernel.
  - $N \in \mathrm{End}(V)$ s.t. $\forall\, \omega \in W_{\mathbb{Q}_p}$, $r(\omega) \circ N \circ r(\omega)^{-1} = |\omega| \cdot N$.

    here, $|\cdot|: W_{\mathbb{Q}_p} \to W_{\mathbb{Q}_p}^{ab} \xrightarrow[\sim]{CFT} \mathbb{Q}_p^\times \xrightarrow{|\cdot|_p} \mathbb{Q}_{>0}$.

Local Langlands is proved for GL_n using ℓ-adic étale cohomology of Lubin-Tate space.

<u>Global Langlands</u>: [OPEN]. Fix prime ℓ and isomorphism $\tau: \overline{\mathbb{Q}_\ell} \xrightarrow{\sim} \mathbb{C}$. There is a bijection:

$$\{\text{"special" cuspidal automorphic repn } \pi \text{ of } GL_2(\mathbb{A})\}$$

$$\updownarrow$$

$$\left\{\begin{array}{l}\text{irr. continuous repn } \rho: \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{Q}_\ell}) \text{ which is unramified almost} \\ \text{everywhere, and } \rho_p: \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \hookrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\rho} GL_2(\overline{\mathbb{Q}_\ell}) \text{ is de Rham for all } p \neq \ell\end{array}\right\}.$$

and it is compatible with Local Langlands.