CAPITAL NORMAL UNIVERSITY, 2026 SPRING
ABSTRACT ALGEBRA II

(1) Find the number of isomorphism classes of groups of order 45.
(2) For a given positive integer n, prove that there are at most $\binom{n!-1}{n-1}$ isomorphism classes of groups of order n.
(3) Let $\mathfrak{p}$ and $\mathfrak{q}$ be two ideals of a commutative ring R. If an ideal I of R satisfies $I \subseteq \mathfrak{p} \cup \mathfrak{q}$, prove that $I \subseteq \mathfrak{p}$ or $I \subseteq \mathfrak{q}$.
(4) If a ring R has m non-zero left zero-divisors and $m > 1$, prove that $\#R \leq (m+1)^2$.
(5) Let $F[0,1]$ be the set of all real-valued functions on the closed interval $[0,1]$. Prove that $F[0,1]$ forms a commutative ring under point-wise addition and multiplication, and that $I = \{f \in F[0,1] : f(0) = 0\}$ is a maximal ideal of F.
(6) Let $n \geq 2$ be a positive integer. Prove that $x^{n-1}y^2 + x^n + 2x^{n-1}y + x^{n-1} + xy + x + y + 1$ is an irreducible element in $\mathbb{Q}[x,y]$.
(7) Let R be a ring, suppose M, N are left R-modules. If $f : M \to N$ and $g : N \to M$ are R-morphisms and $g \circ f : M \to M$ is an automorphism, prove that $N = \ker(g) \oplus \operatorname{im}(f)$.
(8) Prove that $\#\mathrm{GL}_n(\mathbb{F}_p)$ is a factor of $(p^n - 1)!$.
(9) Let R be a principal ideal domain. Prove that every prime ideal in $R[x]$ can be generated by at most two elements.
(10) If F/K is an algebraic extension and D is a subring such that $K \subseteq D \subseteq F$, prove that D is a field. Does this conclusion still hold if F/K is not algebraic?
(11) Let R be a ring, and $n_0 > 1$ be a positive integer. Assume that for any $x \in R$, $x^{n_0} = 0$ implies $x = 0$. Prove that if $x_1, \cdots, x_n \in R$ satisfy $x_1 x_2 \cdots x_n = 0$, then $x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)} = 0$ for any permutation $\sigma \in S_n$.
(12) Let u be an algebraic number over $\mathbb{Q}$, and let $x^3 + 2x - 1$ be its minimal polynomial over $\mathbb{Q}$. Find the minimal polynomial of $u^2 + u$ over $\mathbb{Q}$.
(13) Let $g(x)$ and $h(x)$ be polynomials over a field K, and let u be a root of $g(x)$ in a field extension L/K. Prove that $g(h(x))$ is irreducible over K if and only if $g(x)$ is irreducible over K and $h(x) - u$ is irreducible over $K(u)$.
(14) Let u be an algebraic element over K, prove that $[K(u) : K] \leq n \cdot [K(u^n) : K]$, where $n > 0$ is an integer.
(15) [Wilson] Let p be a prime number. Prove that $(p-1)! \equiv -1 \pmod{p}$.
(16) Let E/F be a finite Galois extension, and let $f \in F[x]$ be the minimal polynomial of $u \in E \backslash F$ over F. Prove that for any two roots r_1, r_2 of f, there exists $\sigma \in \mathrm{Gal}(E/F)$ such that $\sigma(r_1) = r_2$.
(17) Let G be a finite abelian group. Prove that there exists a finite extension $F/\mathbb{Q}$ such that $\mathrm{Gal}(F/\mathbb{Q}) \cong G$. [Hint: if a and d are coprime positive integers, then there are infinitely many primes p such that $p \equiv a \pmod{d}$]
(18) Let E/F be a finite separable extension. Prove that E/F is Galois if and only if any irreducible polynomial in $F[x]$ factors into a product of irreducible polynomials of the same degree in $E[x]$.
(19) Let $\mathbb{Q}(u)/\mathbb{Q}$ be a finite extension, and let $L/\mathbb{Q}$ be the Galois closure of $\mathbb{Q}(u)/\mathbb{Q}$. If p divides $[L : \mathbb{Q}]$, prove that there exists a subfield F of L such that $[L : F] = p$ and $L = F(u)$.
(20) Let E/F be a finite Galois extension and let $\mathrm{Gal}(E/F)$ be a cyclic group. Prove that there is an isomorphism $E^\times / F^\times \cong \ker(N_{E/F})$, where $N_{E/F} : E^\times \to F^\times$ is the norm map.
(21) Let $F \subseteq \mathrm{Mat}_n(\mathbb{Q})$ be a field. Prove that $[F : \mathbb{Q}I_n] \leq n$, where $\mathbb{Q}I_n := \{r \cdot I_n : r \in \mathbb{Q}\}$.