Some Problems on Algebraic Dynamics

Ziyang ZHU [1]

In the summer of 2022, Professor Junyi XIE invited me to introduce algebraic dynamics and entropy theory at BICMR of Peking University, this is the note of this short course.

1 Introduction to Triangulated Categories

The concept of triangulated structure comes from the generalization of exact sequences in the abelian category. In the general cases like Top, the category may not have kernels or cokernels, but they actually have homology or cohomology theory. Hence, there may be a substitute for exactness and then we have homological algebra. We want to construct a suitable category, named the derived category, to reasonably define these possible homological algebras.

In this section, let $\mathscr{A}$ be an additive category, denote $C(\mathscr{A})$ to be the category of complexes of $\mathscr{A}$. Let us construct the derived category of $\mathscr{A}$.

Step 1. Homotopy Category of $C(\mathscr{A})$.

Define $\mathrm{Htp}(X^\cdot, Y^\cdot) := \{f^\cdot : X^\cdot \to Y^\cdot \mid f^\cdot \simeq 0\}$, this is a subgroup of the abelian group $\mathrm{Hom}_{C(\mathscr{A})}(X^\cdot, Y^\cdot)$. Now we define an additive category $K(\mathscr{A})$, called the **homotopy category** of $C(\mathscr{A})$ to be:

$$\text{Objects: objects in } C(\mathscr{A}); \quad \text{Morphisms: } \mathrm{Hom}_{K(\mathscr{A})}(X^\cdot, Y^\cdot) := \mathrm{Hom}_{C(\mathscr{A})}(X^\cdot, Y^\cdot)/\mathrm{Htp}(X^\cdot, Y^\cdot).$$

Step 2. Natural Triangulated Structure on $K(\mathscr{A})$.

Definition 1.1 (Triangle) *Let $[1] : \mathscr{A} \to \mathscr{A}$ be an identity functor. A **triangulated structure** is a class of **distinguished triangles** $\triangle := \left\{ X \stackrel{u}{\to} Y \stackrel{v}{\to} Z \stackrel{w}{\to} X[1] \,\middle|\, X, Y, Z \in \mathscr{A}, u, v, w \in \mathrm{Mor}(\mathscr{A}) \right\}$, which satisfy the following axioms:*

(TR1). $X \stackrel{\mathrm{id}_X}{\to} X \to 0 \to X[1] \in \triangle$.

(TR2). *For any morphism $u : X \to Y$, then $X \stackrel{u}{\to} Y \to Z \to X[1] \in \triangle$ for some Z.*

(TR3). *If $X \stackrel{u}{\to} Y \stackrel{v}{\to} Z \stackrel{w}{\to} X[1] \in \triangle$, then $Y \stackrel{v}{\to} Z \stackrel{w}{\to} X[1] \stackrel{-u[1]}{\to} Y[1] \in \triangle$.*

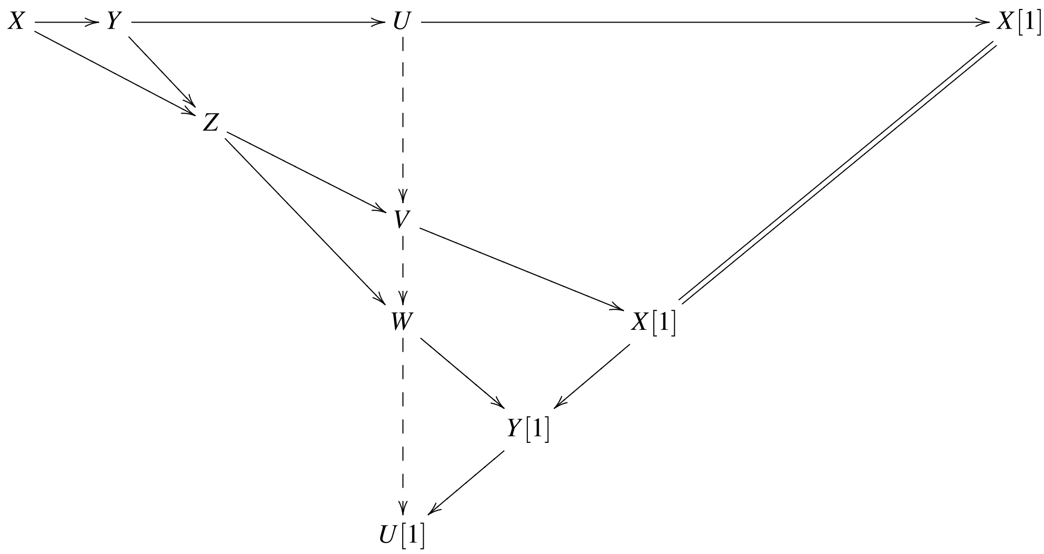
(TR4). *If the rows in the following diagram are all distinguished triangles and the left square is commute:*

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1].
\end{array}$$

[1] Capital Normal University, E-mail: zhuziyang@cnu.edu.cn

then there exists (not unique) a morphism $h : Z \to Z'$ make the whole diagram commute.

(TR5). *If the filled arrows in the following commutative diagram are all distinguished triangles, so is the dashed one.*



The axiom (TR5) is the analogy of taking quotient: if we write $U = Y/X, V = Z/X$, then $W = V/U = (Z/X)/(Y/X) = Z/Y$.

Triangulated structure allows us to do (co)homology theory formally. For example, let $(\mathscr{A}, [1], \triangle)$ be a **triangulated category**, i.e. an additive category together with an identity functor and a triangulated structure with respect to this functor. We call an additive functor H from $\mathscr{A}$ to an abelian category $\mathscr{B}$ a **cohomological functor**, if for any $X \to Y \to Z \to X[1] \in \triangle$, there is a long exact sequence:

$$\cdots \to H(T^i X) \to H(T^i Y) \to H(T^i Z) \to H(T^{i+1} X) \to \cdots.$$

The triangles defined before have a topological interpretion.

Remark 1.2 *Let Y be a topological space and X be its subspace. Consider the following sequence:*

$$X \overset{i}{\hookrightarrow} Y \overset{j}{\hookrightarrow} \mathrm{Cone}(i) \simeq Y/X \hookrightarrow \mathrm{Cone}(j) \simeq \Sigma X := X[1] \to \cdots,$$

where $\mathrm{Cone}(i)$ *is the **topological mapping cone** of i, ΣX is the **suspension** of X. By iteration and taking the i-th singular homology, using the formula $H_i(\Sigma X, \mathbb{Z}) \cong H_{i-1}(X, \mathbb{Z})$ we get the following exact sequence of abelian groups:*

$$\cdots \to H_i(X,\mathbb{Z}) \to H_i(Y,\mathbb{Z}) \to H_i(Y/X,\mathbb{Z}) \to H_{i-1}(X,\mathbb{Z}) \to H_{i-1}(Y,\mathbb{Z}) \to H_{i-1}(Y/X,\mathbb{Z}) \to \cdots.$$

The most important example of triangulated category is the homotopy category defined in Step 1.

Example 1.3 *$K(\mathscr{A})$ has a natural triangulated structure:*

$[1]:\left(X^\cdot:\cdots\to X^i\xrightarrow{d^i}X^{i+1}\to\cdots\right)\mapsto\left(X^\cdot[1]:\cdots\to X^{i+1}\xrightarrow{-d^{i+1}}X^{i+2}\to\cdots\right);(f^\cdot:X^\cdot\to Y^\cdot)\mapsto(f^\cdot[1])^\cdot,$ *where $(f^\cdot[1])^i=f^{i+1}$ for all i.*

Let $u^\cdot:X^\cdot\to Y^\cdot$ be a chain map in $K(\mathscr{A})$, define a complex, called an ***algebraic mapping cone*** *to be:*

$$\left(\mathrm{Cone}(u^\cdot):=X^\cdot[1]\oplus Y^\cdot,\begin{pmatrix}-d_{X^\cdot} & 0\\ u^\cdot & d_{Y^\cdot}\end{pmatrix}\right).$$

This is the algebraic representation of a topological mapping cone in 1.2. Sometimes we also call the diagram $X^\cdot\xrightarrow{u^\cdot}Y^\cdot\xrightarrow{(0,1)}\mathrm{Cone}(u^\cdot)\xrightarrow{(1,0)}X^\cdot[1]$ a ***cone*** *of $u^\cdot$. We say a diagram of complexes $U^\cdot\to V^\cdot\to W^\cdot\to U^\cdot[1]$ is* ***tri-isomorphic*** *to some cone, if the following diagram commutes (homotopically!) for a cone of some chain map.*

$$\begin{array}{ccccccc}
U^\cdot & \longrightarrow & V^\cdot & \longrightarrow & W^\cdot & \longrightarrow & U^\cdot[1]\\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong\\
X^\cdot & \xrightarrow{u^\cdot} & Y^\cdot & \longrightarrow & \mathrm{Cone}(u^\cdot) & \longrightarrow & X^\cdot[1]
\end{array}$$

Now define a triangulated structure on $K(\mathscr{A})$:

$\triangle_{\mathscr{A}}:=\{$*diagrams tri-isomorphic to some cone*$\}$.

Step 3. Localization of $K(\mathscr{A})$ with Respect to Some Localizing Class.

Theorem 1.4 (Localization) *Let $\mathscr{A}$ be a category and S be an arbitrary class of morphisms in $\mathscr{A}$. Then there exists a category $S^{-1}\mathscr{A}$, called* ***the localization of $\mathscr{A}$ with respect to*** *S, and a* ***localization functor*** *$Q:\mathscr{A}\to S^{-1}\mathscr{A}$, which satisfies the following properties:*

(1). For any $s\in S$, $Q(s)$ is an isomorphism in $S^{-1}\mathscr{A}$.

(2). For any category $\mathscr{B}$ and any functor $F:\mathscr{A}\to\mathscr{B}$ satisfying $F(s)$ is an isomorphism for all $s\in S$, there exists a unique functor $G:S^{-1}\mathscr{A}\to\mathscr{B}$ make the following diagram commutes.

$$\begin{array}{ccc}
\mathscr{A} & \xrightarrow{Q} & S^{-1}\mathscr{A}\\
 & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle G} & \\
 & \mathscr{B} &
\end{array}$$

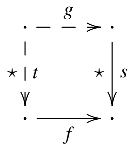
It is very difficult to find the localization in the Theorem in general. However, in the special case we introduce below, when S is a localizing class, the structure of $S^{-1}\mathscr{A}$ is clear.

Definition 1.5 (Localizing Class) *A class of morphisms S in an abelian category $\mathscr{A}$ is a* ***localizing class****, if it has the following properties:*

(LC1). *For any $M \in \mathscr{A}$, $\mathrm{id}_M \in S$.*

(LC2). *If $s,t \in S$, then $s \circ t \in S$ (if it can be composited).*

(LC3). *For any $f \in \mathrm{Mor}(\mathscr{A})$ and $s \in S$, there exist two morphisms (not unique) $g \in \mathrm{Mor}(\mathscr{A})$ and $t \in S$, make the following diagram commutes:*

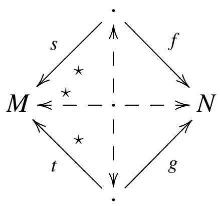


The statement of reversing the above arrow is satisfied dually. Here we emphasize "the morphism in S" with the sign "$\star$".

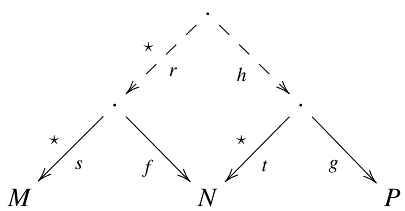
(LC4). *Let $f,g : M \to N$ be two morphisms in $\mathscr{A}$, then $s \circ f = s \circ g$ for some $s \in S$ if and only if $f \circ t = g \circ t$ for some $t \in S$.*

The morphisms in the localization are the so-called "roofs".

Definition 1.6 (Roof) *Let $\mathscr{A}$ be a category and S be a localizing class in $\mathscr{A}$. A **roof** between M and N is a diagram $M \xleftarrow[\star]{s} \cdot \xrightarrow{f} N$. We say two roofs $M \xleftarrow[\star]{s} \cdot \xrightarrow{f} N$ and $M \xleftarrow[\star]{t} \cdot \xrightarrow{g} N$ are **equivalent**, if there is a commutative diagram:*



*This is an equivalence relation. Define the **composition** of two roofs $M \xleftarrow[\star]{s} \cdot \xrightarrow{f} N$ and $N \xleftarrow[\star]{t} \cdot \xrightarrow{g} P$ to be the diagram $M \xleftarrow[\star]{s \circ r} \cdot \xrightarrow{h \circ g} P$ comes from* (LC2) *and* (LC3)*:*



This does not depend on the selection of representative elements and the way the diagram is completed.

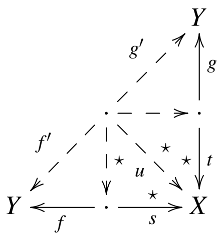
Remark 1.7 *The concept of roofs also have topological interpretion. In homotopy theory, Whitehead theorem reminds us to invert morphisms in S.*

Definition 1.8 (Addition) *Let S be a localizing class of an additive category $\mathscr{A}$. Define a category*

$(\mathscr{A}, S)$ to be:

$$\text{Objects: objects in } \mathscr{A}; \quad \text{Morphisms: } \mathrm{Hom}_{(\mathscr{A},S)}(X,Y) := \{\text{roofs between } X \text{ and } Y\}/\sim.$$

Define the abelian group structure on $\mathrm{Hom}_{(\mathscr{A},S)}(X,Y)$ as $X \underset{\star}{\overset{s}{\longleftarrow}} \cdot \overset{f}{\longrightarrow} Y + X \underset{\star}{\overset{t}{\longleftarrow}} \cdot \overset{g}{\longrightarrow} Y := X \underset{\star}{\overset{u}{\longleftarrow}} \cdot \overset{f'+g'}{\longrightarrow} Y$ in the following diagram:



Hence $(\mathscr{A}, S)$ is an additive category.

Proposition 1.9 *$S^{-1}\mathscr{A}$ and $(\mathscr{A}, S)$ are equivalent categories, and the functor in 1.4 is given by $Q : \mathscr{A} \to (\mathscr{A}, S) \cong S^{-1}\mathscr{A}, f \mapsto \left(\cdot \underset{\star}{\overset{\mathrm{id}}{\longleftarrow}} \cdot \overset{f}{\longrightarrow} \cdot\right)$.*

Example 1.10 *Let $\mathscr{A}$ be an additive category and $K(\mathscr{A})$ be its homotopy category. We call a morphism in $K(\mathscr{A})$ a **quasi-isomorphism**, if all of its representations are quasi-isomorphisms in $C(\mathscr{A})$. It is easy to check that $S_{\mathscr{A}} := \{\text{all quasi-isomorphisms in } K(\mathscr{A})\}$ is a localizing class in $K(\mathscr{A})$. By 1.9, $(K(\mathscr{A}), S_{\mathscr{A}}) \cong S_{\mathscr{A}}^{-1}K(\mathscr{A})$ is the localization of $K(\mathscr{A})$ with respect to $S_{\mathscr{A}}$.*

Step 4. Induced Triangulated Structure on Localization.

Definition 1.11 (Compatible) *Let $(\mathscr{A}, [1], \triangle)$ be a triangulated category and S be a localizing class in $\mathscr{A}$. We say S is **compatible** with $\triangle$, if:*

(CP1). *$s \in S$ if and only if $s[1] \in S$.*

(CP2). *The morphisms h in* (TR4) *are also in S.*

Proposition 1.12 *Let $(\mathscr{A}, [1], \triangle)$ be a triangulated category and S be a localizing class compatible with $\triangle$. Then $S^{-1}\mathscr{A}$ is also a triangulated category with triangulated structure*

$$Q\triangle := \left\{X \overset{Qu}{\to} Y \overset{Qv}{\to} Z \overset{Qw}{\to} X[1] \middle| X \overset{u}{\to} Y \overset{v}{\to} Z \overset{w}{\to} X[1] \in \triangle\right\}.$$

Example 1.13 *Let $\mathscr{A}$ be an additive category, in 1.10 we get an additive category $S_{\mathscr{A}}^{-1}K(\mathscr{A})$. Now define the **derived category** of $\mathscr{A}$ or $C(\mathscr{A})$ to be a triangulated category $D(\mathscr{A}) := (S_{\mathscr{A}}^{-1}K(\mathscr{A}), [1], Q\triangle_{\mathscr{A}})$.*

Remark 1.14 *Let $\mathscr{A}$ be an abelian category, then $D(\mathscr{A})$ not necessarily an abelian category. The distinguished triangles in $D(\mathscr{A})$ are come from the short exact sequences in $\mathscr{A}$, by considering the complex only nontrivial at degree 0 and regarding all quasi-isomorphisms as isomorphisms. Hence, the way of* *selecting complexes in the derived category is canonical, so the target of shift functor* $[1]$ *on complexes is well-defined. In 1.3, use* (TR3) *for a cone we find that the natural triangulated structure implies the short exact sequences are split, so in fact,* $D(\mathscr{A})$ *is abelian if and only if all short exact sequences in* $\mathscr{A}$ *are split.*

Remark 1.15 *Sometimes we use* D^+ *(resp.* D^-, D^b, K^+, K^-, K^b*) to represent the lower-bounded derived category (resp. upper-bounded derived category, bounded derived category, lower-bounded homotopy category, upper-bounded homotopy category, bounded homotopy category).*

Step 5. Derived Functors from Derived Category as a Triangulated Category.

Let $F : \mathscr{A} \to \mathscr{B}$ be a left (resp. right) exact functor between abelian categories. The right (resp. left) derived functor of F is something like $RF : D^+(\mathscr{A}) \to K^+(\mathrm{Inj}(\mathscr{A})) \xrightarrow{F} K^+(\mathscr{B}) \xrightarrow{Q} D^+(\mathscr{B})$ $\big($resp. $LF : D^-(\mathscr{A}) \to K^-(\mathrm{Proj}(\mathscr{A})) \xrightarrow{F} K^-(\mathscr{B}) \xrightarrow{Q} D^-(\mathscr{B})\big)$. Now let us take the left exact functor as an example to establish this theory.

Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor between abelian categories. Define a functor $\widetilde{F} : C(\mathscr{A}) \to C(\mathscr{A}), \big(X^\cdot : \cdots \to X^i \xrightarrow{d^i} X^{i+1} \to \cdots\big) \mapsto \big(FX^\cdot : \cdots \to FX^i \xrightarrow{Fd^i} FX^{i+1} \to \cdots\big); (f^\cdot : X^\cdot \to Y^\cdot) \mapsto (Ff^\cdot : FX^\cdot \to FY^\cdot)$. Then we have the following diagram:

$$\begin{array}{ccccccc}
\mathscr{A} & \xhookrightarrow{X \mapsto (\cdots \to 0 \to X \to 0 \to \cdots)} & C^+(\mathscr{A}) & \longrightarrow & K^+(\mathscr{A}) & \xrightarrow{Q} & D^+(\mathscr{A}) \\
\downarrow F & & \downarrow \widetilde{F} & & \downarrow \overline{F} & & \downarrow \widehat{F} \\
\mathscr{B} & \hookrightarrow & C^+(\mathscr{B}) & \longrightarrow & K^+(\mathscr{B}) & \xrightarrow{Q} & D^+(\mathscr{B})
\end{array}$$

Here $\overline{F}$ is a **triangulated functor**, i.e. $\overline{F}$ commutes with $[1]$ and $\overline{F}\triangle_{\mathscr{A}} \subseteq \triangle_{\mathscr{B}}$.

The triangulated functor $\widehat{F}$ which makes the diagram commutes (homotopically!) may not exist in general. In fact, $\widehat{F}$ exists if and only if for any quasi-isomorphism f, $\widetilde{F}f$ is also a quasi-isomorphism.

The right derived functor of F is a certain $\widehat{F}$ satisfying a universal property.

Definition 1.16 (Derived Functor) *The* ***right derived functor*** (RF, ε) *of* F *is a triangulated functor* $RF : D^+(\mathscr{A}) \to D^+(\mathscr{B})$ *together with a natural transformation* $\varepsilon : Q \circ \overline{F} \to RF \circ Q$ *have the universal property: for any triangulated functor* $G : D^+(\mathscr{A}) \to D^+(\mathscr{B})$ *and any natural transformation* $\eta : Q \circ \overline{F} \to G \circ Q$*, there exists a unique natural transformation* $\alpha : RF \to G$*, such that* $\eta = \alpha \circ \varepsilon$*.*

If (RF, ε) exists, it is unique up to a unique isomorphism.

Theorem 1.17 *Let* $F : \mathscr{A} \to \mathscr{B}$ *be an additive functor. Assume that there exists a full additive subcategory* $\mathscr{A}'$ *of* $\mathscr{A}$*, such that:*

(i). For all $X \in \mathscr{A}$*, there exists* $X' \in \mathscr{A}$*, and a monomorphism* $X \hookrightarrow X'$*.*

(ii). If $0 \to X \to Y \to Z \to 0$ *is an exact sequence, and* $X, Y \in \mathscr{A}'$*, then* $Z \in \mathscr{A}'$*, and* $0 \to FX \to FY \to FZ \to 0$ *is exact. Then* $RF : D^+(\mathscr{A}) \to D^+(\mathscr{B})$ *exists and for* $W \in D^+(\mathscr{A})$*, we have an isomorphism* $RFW \xleftarrow{\sim} FW'$*, where* $W \to W'$ *is a quasi-isomorphism with* $W' \in K^+(\mathscr{A}')$*.*

So if $\mathscr{A}$ has enough injective objects, then $RF: D^+(\mathscr{A}) \to D^+(\mathscr{B})$ exists, and for any $W \in D^+(\mathscr{A})$, if $W \to W'$ is a quasi-isomorphism with $W' \in K^+(\mathscr{A})$ and W' consists of injective objects, then $FW' \xrightarrow{\sim} RFW$.

Example 1.18 *(i).* $\mathrm{Hom}(X,\cdot) \Rightarrow \mathrm{Ext}^n(X,\cdot)$ *or* $\mathrm{Hom}(X,\cdot[n]) = H^n R\mathrm{Hom}(X,\cdot)$; $\mathscr{E}\mathrm{xt}^{\cdot}, Rf_*$ *are similar.*

(ii). $X \otimes \cdot \Rightarrow \mathrm{Tor}_n(X,\cdot)$ *or* $X \otimes (\cdot[-n]) = H_n(X \otimes^L \cdot)$; Lf^* *is similar.*

(iii). $D^b(\mathrm{Coh}(\mathbb{P}^1)) \xrightarrow{\sim} D^b(\mathrm{Rep}(\cdot \rightrightarrows \cdot))$. *See the remark below.*

Let $\langle\cdot\rangle$ denote the smallest triangulated subcategory containing this object closed under isomorphisms and taking direct summands.

Remark 1.19 *In some cases, the triangulated category can be generated by some special objects. For example, in 1.18(iii), use representation theory we can prove $D^b(\mathrm{Coh}(\mathbb{P}^1)) = \langle \mathscr{O} \oplus \mathscr{O}(1)\rangle$, due to the theorem of Beilinson-Bondal, see [AABE].*

This sheaf is a tilting object (also a split generator, see 2.2) in $D^b(\mathrm{Coh}(\mathbb{P}^1))$. A ***tilting object*** *for a non-singular projective variety X is a coherent sheaf $\mathscr{T}$ with:*

(i). $\mathrm{Ext}^i(\mathscr{T},\mathscr{T}) = 0$ *for all* $i > 0$.

(ii). $\mathscr{T}$ *generates* $D^b(\mathrm{Coh}(X))$ *as a triangulated category, i.e.* $D^b(\mathrm{Coh}(X)) = \langle \mathscr{T} \rangle$.

(iii). The algebra $\Lambda := \mathrm{End}(\mathscr{T})^{\mathrm{op}}$ *has finite global dimension.*

The tilting objects give the following equivalence of categories:

$$R\mathrm{Hom}(\mathscr{T},\cdot): D^b(\mathrm{Coh}(X)) \longrightarrow D^b(\mathrm{Mod}(\Lambda)); \quad \mathscr{T} \otimes^L \cdot : D^b(\mathrm{Mod}(\Lambda)) \longrightarrow D^b(\mathrm{Coh}(X)).$$

Proposition 1.20 *$\mathscr{O} \oplus \mathscr{O}(1)$ is a tilting sheaf for $\mathbb{P}^1$, and $\Lambda = (\cdot \rightrightarrows \cdot)$. In general, the tilting object for $\mathbb{P}^n$ is $\bigoplus_{i=0}^{n} \mathscr{O}(i)$.*

PROOF. The vanishing of Extension groups are given by Serre duality, or in fact we have a equation $\dim_{\mathbb{C}} \mathrm{Ext}^1(\mathscr{O}(i),\mathscr{O}(j)) = \dim \mathbb{C}[x,y]_{i-j-2}$. The finiteness is given by glo. dim.(path algebra) = 1. Since the sequences $0 \to \mathscr{O}(i) \to \mathscr{O}(i+1)^2 \to \mathscr{O}(i+2) \to 0$ are exact for all i, use (TR2) we obtain $\mathscr{O}(2) \in \langle \mathscr{O} \oplus \mathscr{O}(1)\rangle$, by induction we can prove the first part of the proposition. □

2 Entropy of Endofunctors

We start this section with the following fact.

Theorem 2.1 (Gromov-Yomdin) *Let X be a compact Kähler manifold, $f: X \to X$ be a surjective holomorphic map. Then the topological entropy $h_{\mathrm{top}}(f) = \log\big(\text{the spectral radius of } Hf: H^{\cdot}(X) \to H^{\cdot}(X)\big)$.*

It's not surprising that the cohomological information may be related to dynamical information. Our question is, how to use the derived category of category of coherent sheaves (cohomological information) on X to express and calculate the topological entropy (dynamical information) of a given map? This leads to the concept of categorical entropy.

Remark 2.2 *Let X be a smooth projective variety over $\mathbb{C}$ and $\dim_{\mathbb{C}} X = d$. Consider the bounded triangulated category $D^b(X) := \big(D^b(\mathrm{Coh}(X)), [1],\ cone\big)$. $D^b(X)$ has some properties:*

(SPLIT CLOSED). *$D^b(X)$ contains all direct summands of its objects.*

(OF FINITE TYPE). $\sum_{n\in\mathbb{Z}} \dim_{\mathbb{C}} \mathrm{Hom}_{D^b(X)}(M, N[n]) < \infty$ *for all $M, N \in D^b(X)$.*

(HAS SPLIT GENERATORS). *There exists a $G \in D^b(X)$, such that $D^b(X) = \langle G\rangle$, where $\langle G\rangle$ is the smallest triangulated subcategory containing G closed under isomorphisms and taking direct summands.*

The condition (HAS SPLIT GENERATORS) is non-trivial. For example, if $\mathscr{L}$ is an ample line bundle on X (X needs projective!), we can take $G = \bigoplus_{i=1}^{d+1} \mathscr{L}^i$ (see 1.20).

Definition 2.3 (Categorical Complexity) *Let M, N be two objects in $D^b(X)$. Define a function $\delta_t(M,N) : \mathbb{R} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$, called the **complexity of** N **with respect to** M to be:*

$$\delta_t(M,N) := \begin{cases} 0 & N \cong 0 \\ \inf\left\{ \sum_{i=1}^{p} e^{n_i t} \;\middle|\; \begin{array}{ccccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \cdots & A_{p-1} & \longrightarrow & N \oplus N' \\ & \nwarrow & \downarrow & \nwarrow & \downarrow & & & & & \nwarrow & \downarrow \\ & & M[n_1] & & M[n_2] & & & & & & M[n_p] \end{array} \right\} & N \in \langle M\rangle \\ \infty & N \notin \langle M\rangle \end{cases}$$

Remark 2.4 *The concept of categorical complexity comes from the **Bridgeland stability condition**. It has a topological interpretion. Let $X = \{*\} \cup \{*'\}$ be a double points space, then $\Sigma^n X = X[n] = S^n$. Suppose Y is another topological space containing X as its subspace.*

(i). If $Y = S^n$, then

$$\begin{array}{ccc} \cdot(0) & \longrightarrow & S^n \\ & \swarrow & \\ & S^n & \end{array}$$

(ii). If $Y = S^1 \vee S^1$, then

$$\begin{array}{ccccc} \cdot(0) & \longrightarrow & S^1 & \longrightarrow & Y \\ & \swarrow & & \swarrow & \\ S^1 & & & S^1 & \end{array}$$

The so-called complexity here is asked how many n-cells does Y have.

Definition 2.5 (Entropy) *Let G be a split generator of $D^b(X)$, and $F : D^b(X) \to D^b(X)$ be a triangulated functor such that $F^n G \ncong 0$ for all $n > 0$ (if $F^n G = 0$ for some n, then $F^n M \cong 0$ for all M). Define the* ***entropy*** *of F is the function*

$$h_t(F) := \lim_{n\to\infty} \frac{1}{n} \log \delta_t(G, F^n G) : \mathbb{R} \to \{-\infty\} \cup \mathbb{R}.$$

For a surjective endomorphism $f : X \to X$, the ***categorical entropy*** *$h_{\text{cat}}(f)$ of f is defined to be $h_0(Lf^*)$. This records the minimal length of diagrams in 2.3.*

Proposition 2.6 *Let G, G' be split generators of $D^b(X)$ and $F : D^b(X) \to D^b(X)$ be a triangulated functor such that $F^n G, F^n G' \ncong 0$ for all $n > 0$. Then $h_t(F) = \lim_{n\to\infty} \frac{1}{n} \log \delta_t(G, F^n G')$.*

The proof of 2.6 (see below) shows that the definition of entropy is independent of the choice of split generators. In the theory of mapping class groups, 2.6 has the following analogy. This example is due to Nielsen-Thurston.

Example 2.7 (Pesudo-Anosov Maps on Riemann Surfaces) *Let X be a Riemann surface with genus ≥ 1, and $f : X \to X$ be a pesudo-anosov λ-map. If X has a suitable Riemann metric g and two loops a, b, then*

$$h_{\text{top}}(f) = \lim_{n\to\infty} \frac{1}{n} \log\big(\textit{geometric intersection number } (a, f^n b)\big) = \lim_{n\to\infty} \frac{1}{n} \log(\text{length}_g(f^n a)) = \log \lambda.$$

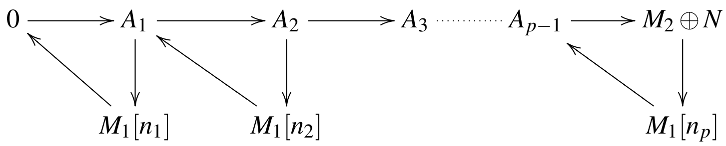
To prove proposition 2.6, we need some lemmas, which give some basic properties of complexity.

Lemma 2.8 *Let $M_1, M_2, M_3 \in D^b(X)$.*
(i). $M_1 \cong M_3 \Rightarrow \delta_t(M_1, M_2) = \delta_t(M_3, M_2)$.
(ii). $M_2 \cong M_3 \Rightarrow \delta_t(M_1, M_2) = \delta_t(M_1, M_3)$.
(iii). $\delta_t(M_1, M_3) \leq \delta_t(M_1, M_2)\delta_t(M_2, M_3)$.
(iv). $\delta_t(M_1, M_2 \oplus M_3) \leq \delta_t(M_1, M_2) + \delta_t(M_1, M_3)$.
(v). If $F : D^b(X) \to D^b(X)$ is a triangulated functor, then $\delta_t(FM_1, FM_2) \leq \delta_t(M_1, M_2)$.

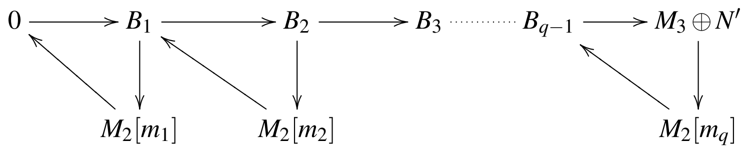
PROOF. Only prove (i) and (iii).
(i). $0 \to A_1 \to M_1[n_1] \to 0$ is tri-isomorphic to $0 \to A_1 \to M_3[n_1] \to 0$:

$$\begin{array}{ccccccc}
0 & \longrightarrow & A_1 & \xrightarrow{f_1} & M_1[n_1] & \longrightarrow & 0 \\
{\scriptstyle 0}\big\downarrow & & \big\downarrow{\scriptstyle =} & & {\scriptstyle \cong}\big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle 0} \\
0 & \longrightarrow & A_1 & \xrightarrow[i\circ f_1]{} & M_3[n_1] & \longrightarrow & 0
\end{array}$$

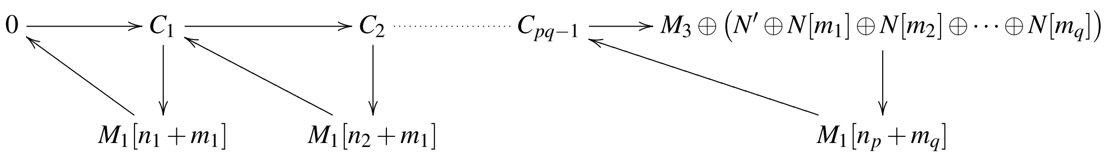
(iii). Given a $\varepsilon > 0$. There exist



such that $\sum_{i=1}^{p} e^{n_i t} < \delta_t(M_1, M_2) + \varepsilon$, and

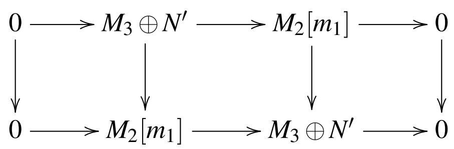


such that $\sum_{j=1}^{q} e^{n_j t} < \delta_t(M_2, M_3) + \varepsilon$. Then we can construct

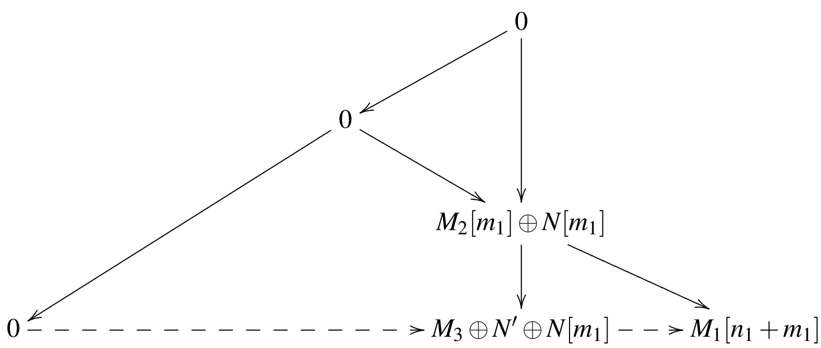


so $\delta_t(M_1, M_3) \leq \sum_{i,j} e^{(n_i + m_j)t} \leq (\delta_t(M_1, M_2) + \varepsilon)(\delta_t(M_2, M_3) + \varepsilon)$.

For example, if we have $0 \to M_2 \oplus N \to M_1[n_1] \to 0$ and $0 \to M_3 \oplus N' \to M_2[m_1] \to 0$ (i.e. $p = q = 1$), then use (TR3) we get $0 \to M_2[m_1] \oplus N[m_1] \to M_1[n_1 + m_1] \to 0$. Of course, we have a tri-isomorphism since quasi-isomorphisms are isomorphisms in the derived category:



Now use (TR5), we have



Hence, (iii) is proved by induction. □

PROOF OF 2.6. Use 2.8(iii) and (v),

$$\delta_t(G,F^nG') \leq \delta_t(G,F^nG)\delta_t(F^nG,F^nG') \leq \delta_t(G,G')\delta_t(G,F^nG).$$

Similarly, $\delta_t(G,F^nG) \leq \delta_t(G',G)\delta_t(G,F^nG')$. Take the limit and get the conclusion. □

Proposition 2.9 *Let $G,G' \in D^b(X)$ be split generators, $F_1,F_2 : D^b(X) \to D^b(X)$ are two triangulated functors such that $F_i^nG, F_i^nG' \ncong 0$ for all $i = 1,2$ and $n > 0$. Then:*

(i). $F_1 \cong F_2 \Rightarrow h_t(F_1) = h_t(F_2)$.

(ii). $h_t(F_1^m) = mh_t(F_1)$ for all $m \in \mathbb{Z}_{\geq 1}$ $\big($this is analogous to $h_{\mathrm{top}}(T^m) = mh_{\mathrm{top}}(T)$, where T is a continuous map on a compact metric space$\big)$.

(iii). $h_t([m]) = mt$ for all $m \in \mathbb{Z}$.

(iv). $F_1F_2 \cong F_2F_1 \Rightarrow h_t(F_1F_2) \leq h_t(F_1) + h_t(F_2)$.

(v). $F_1 = F_2 \circ [m] \Rightarrow h_t(F_1) = h_t(F_2) + mt$.

(vi). $h_t(\mathscr{L} \otimes \cdot) = 0$ for all $\mathscr{L} \in \mathrm{Pic}(X)$.

*(vii). If F_1 is of **Fourier-Mukai type**, i.e. $F_1(\cdot) \cong R\pi_{2*}(P \otimes^L_{X\times_{\mathbb{C}}X} L\pi_1^*(\cdot))$ for some $P \in D^b(X \times_{\mathbb{C}} X)$.*

$$\begin{array}{ccc} X \times_{\mathbb{C}} X & \xrightarrow{\pi_1} & X \\ {\scriptstyle \pi_2}\downarrow & & \downarrow \\ X & \longrightarrow & \mathrm{Spec}(\mathbb{C}) \end{array}$$

Then $h_t(F_1) = \lim\limits_{n\to\infty} \dfrac{1}{n} \log \left(\sum\limits_{m\in\mathbb{Z}} \dim_{\mathbb{C}} \mathrm{Hom}_{D^b(X)}(G, F_1^nG'[m]) \cdot e^{-mt} \right).$

PROOF. We only prove (iii) and (vii). Use (vii) we obtain (vi).

(iii). Since we have a distinguished triangle $0 \to G[nm] \to G[nm] \to 0$, this implies $\delta_t(G,G[nm]) = e^{nmt}$. Hence, $h_t([m]) = \lim\limits_{n\to\infty} \frac{1}{n} \log \delta_t(G,G[nm]) = \lim\limits_{n\to\infty} \frac{1}{n} nmt = mt$.

(vii). Write $\widetilde{\delta_t}(M,N) := \sum\limits_{m\in\mathbb{Z}} \dim_{\mathbb{C}} \mathrm{Hom}_{D^b(X)}(M,N[m]) \cdot e^{-mt}$. Formally, for any $M \in D^b(X)$, apply 2.8 we have

$$\delta_t(\mathbb{C}, R\mathrm{Hom}(G,M)) \leq \delta_t(\mathbb{C}, R\mathrm{Hom}(G,G))\delta_t(R\mathrm{Hom}(G,G), R\mathrm{Hom}(G,M)) \leq \delta_t(\mathbb{C}, R\mathrm{Hom}(G,G))\delta_t(G,M).$$

Hence, $\delta_t(\mathbb{C}, R\mathrm{Hom}(G,M)) \leq C_2(t)\delta_t(G,M)$, where $C_2(t) : \mathbb{R} \to \mathbb{R}_{>0}$ independent of M. On the other hand, use the property of saturated A_∞-categories (see [DHKK, section 2]) and assume $M = F_1^nG'$, we have

$$C_1\delta_t(G,F_1^nG') \leq \delta_t(\mathbb{C}, R\mathrm{Hom}(G,F_1^nG')) = \sum_m \dim_{\mathbb{C}} \mathrm{Ext}^m(G,F_1^nG')e^{-mt} = \widetilde{\delta_t}(G,F_1^nG') \leq C_2(t)\delta_t(G,F_1^nG').$$

Now use 2.6, we obtain $h_t(F_1) = \lim_{n\to\infty} \frac{1}{n} \log \widetilde{\delta}_t(G, F_1^n G')$. $\square$

Let X be a d-dimensional complex smooth projective variety, and f be a surjective endomorphism of X. For an integer $q (0 \le q \le d)$, define $H^{q,q}(X,\mathbb{R}) := H^{q,q}(X) \cap H^{2q}(X,\mathbb{R})$. By Hodge theory, we have $H^{q,q}(X,\mathbb{R}) \otimes \mathbb{C} = H^{q,q}(X)$. Note that f induces an automorphism $f^* : H^{q,q}(X,\mathbb{R}) \to H^{q,q}(X,\mathbb{R})$.

Definition 2.10 (Dynamical Degree) *Define the q-th* ***dynamical degree*** $d_q(f)$ *of f by*

$$d_q(f) := \limsup_{n\to\infty} \left(\int_X \left[(f^n)^* c_1(\mathscr{L})^q \right] \wedge c_1(\mathscr{L})^{d-q} \right)^{\frac{1}{n}},$$

where $\mathscr{L}$ is a very ample invertible sheaf on X. It is not difficult to check that the q-th dynamical degree does not depend on the choice of $\mathscr{L}$.

The dynamical degrees measure the complexity of f in a sense. When f is easy, we hope $d_q(f) = 1$, or they are small enough.

Decompose $H^{q,q}(X)$ into the direct sum of the f^*-invariant subspaces:

$$H^{q,q}(X) = H^{q,q}(X)_{\lambda_1,m_1} \oplus \cdots \oplus H^{q,q}(X)_{\lambda_s,m_s}, \quad (|\lambda_1|, m_1) \ge \cdots \ge (|\lambda_s|, m_s),$$

where $f^*|_{H^{q,q}(X)_{\lambda_i,m_i}}$ is given by the Jordan block J_{λ_i,m_i} (λ_i are eigenvalues, m_i are orders of matrices) and we write $(|\lambda_i|, m_i) \ge (|\lambda_j|, m_j)$ if either $|\lambda_i| > |\lambda_j|$ or $|\lambda_i| = |\lambda_j|$ and $m_i \ge m_j$.

In order to make a connection with 2.1, we define:

Definition 2.11 *Let the notations as above. The* ***spectral radius*** $r_q(f)$ *of the automorphism f^* : $H^{q,q}(X,\mathbb{R}) \to H^{q,q}(X,\mathbb{R})$ is the positive number $|\lambda_i|$, the maximum of absolute values of eigenvalues of f^*. The* ***multiplicity*** $l_q(f)$ *of the spectral radius $r_q(f)$ is the integer m_i.*

Note that for any matrix norm $\|\cdot\|$, we have

$$\limsup_{n\to\infty} \frac{\|J^n_{\lambda_i,m_i}\|}{n^{m_i-1}\lambda_i^n} < \infty; \quad \limsup_{n\to\infty} \frac{n^{m_i-1}\lambda_i^n}{\|J^n_{\lambda_i,m_i}\|} < \infty.$$

The following proposition is an analogy of this fact.

Proposition 2.12 *Let $\mathscr{L}$ be a very ample invertible sheaf on X. For an integer $q(0 \le q \le d)$,*

$$\limsup_{n\to\infty} \frac{\int_X (f^*)^n c_1(\mathscr{L})^q \wedge c_1(\mathscr{L})^{d-q}}{n^{l_q(f)-1} r_q(f)^n} < \infty; \quad \limsup_{n\to\infty} \frac{n^{l_q(f)-1} r_q(f)^n}{\int_X (f^*)^n c_1(\mathscr{L})^q \wedge c_1(\mathscr{L})^{d-q}} < \infty.$$

In particular, use these two limits we can find that $\left(\int_X \left[(f^n)^* c_1(\mathscr{L})^q \right] \wedge c_1(\mathscr{L})^{d-q} \right)^{\frac{1}{n}}$ converges to the q-th dynamical degree $d_q(f)$, which coincides with the spectral radius $r_q(f)$.

2.12 implies that the spectral radius of Hf is equal to $\max_q\{r_q(f)\} = \max_q\{d_q(f)\}$.

The next theorem is one of the main theorems in the theory of categorical entropy:

Theorem 2.13 (Kikuta-Takahashi) *Let X be a smooth projective variety over $\mathbb{C}$. For each surjective endomorphism f,*

$$h_{\text{cat}}(f) = \lim_{n\to\infty}\frac{1}{n}\log\left|\sum_{m\in\mathbb{Z}}(-1)^m\dim_{\mathbb{C}}\operatorname{Hom}_{D^b(X)}(G,(Lf^*)^nG^\sharp[m])\right|,$$

here $G^\sharp = \bigoplus_{i=1}^{d+1}\mathscr{L}^{-i}$ is the duality of $G = \bigoplus_{i=1}^{d+1}\mathscr{L}^i$ and it is also a split generator. In particular, we have $h_{\text{cat}}(f) = h_{\text{top}}(f)$.

PROOF SKETCH. Note that Lf^* is of Fourier-Mukai type, so by 2.9(vii) we have

$$h_{\text{cat}}(f) = h_0(Lf^*) = \lim_{n\to\infty}\frac{1}{n}\log\widetilde{\delta}_0(G,(Lf^*)^nG^\sharp) = \lim_{n\to\infty}\frac{1}{n}\log|\chi(G,(Lf^*)^nG^\sharp)|,$$

where $\chi(M,N) := \sum(-1)^n\dim_{\mathbb{C}}\operatorname{Hom}_{D^b(X)}(M,N[n])$, since Kodaira vanishing theorem implies $\widetilde{\delta}_0(G,(Lf^*)^nG^\sharp) = (-1)^d\chi(G,(Lf^*)^nG^\sharp) = |\chi(G,(Lf^*)^nG^\sharp)|$. Now apply Grothendieck-Riemann-Roch,

$$\chi(G,(Lf^*)^nG^\sharp) = \int_X \operatorname{ch}(G^\sharp)\operatorname{ch}[(f^*)^nG^\sharp]\operatorname{td}(X).$$

It follows from explicit calculations that

$$(-1)^d\int_X \operatorname{ch}(G^\sharp)\operatorname{ch}[(f^*)^nG^\sharp]\operatorname{td}(X) = \sum_{r=0}^{d}\sum_{q=0}^{d-r}C_{r,q}\int_X\left[(f^*)^nc_1(\mathscr{L})^q\right]\wedge c_1(\mathscr{L})^{d-r-q}\operatorname{td}_r(X).$$

Let p be an integer such that $(r_p(f),l_p(f)) = \max_q\{(r_q(f),l_q(f))\}$, We have

$$h_{\text{cat}}(f) = \log d_p(f) + \lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{r=0}^{d}\sum_{q=0}^{d-r}C_{r,q}\frac{\int_X[(f^*)^nc_1(\mathscr{L})^q]\wedge c_1(\mathscr{L})^{d-r-q}\operatorname{td}_r(X)}{\int_X[(f^*)^nc_1(\mathscr{L})^p]\wedge c_1(\mathscr{L})^{d-p}}\right).$$

Recall that $\int_X[(f^*)^nc_1(\mathscr{L})^q]\wedge c_1(\mathscr{L})^{d-r-q}\operatorname{td}_r(X)$ has at most the growth $n^{l_q(f)-1}r_q(f)^n$ as $n\to\infty$ by 2.12. Hence,

$$\limsup_{n\to\infty}\left|\frac{\int_X[(f^*)^nc_1(\mathscr{L})^q]\wedge c_1(\mathscr{L})^{d-r-q}\operatorname{td}_r(X)}{\int_X[(f^*)^nc_1(\mathscr{L})^p]\wedge c_1(\mathscr{L})^{d-p}}\right| < \infty.$$

Note that $h_{\text{cat}}(f)$ and $\log d_p(f)$ do not depend on the choice of $\mathscr{L}$. It follows that

$$\lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{r=0}^{d}\sum_{q=0}^{d-r}C_{r,q}\frac{\int_X[(f^*)^nc_1(\mathscr{L})^q]\wedge c_1(\mathscr{L})^{d-r-q}\operatorname{td}_r(X)}{\int_X[(f^*)^nc_1(\mathscr{L})^p]\wedge c_1(\mathscr{L})^{d-p}}\right) = 0.$$

Hence, combined with 2.1 and 2.12 we have $h_{\text{cat}}(f) = \log d_p(f) = \log r_p(f) = \log(\rho(Hf)) = h_{\text{top}}(f)$. □

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